# Geometric Numerical Integration
# Homework - Lie-group methods with frozen flows

by Adrien Laurent

The homework is due for November 30, 2024. This assignment is not mandatory for completing the course, though the final grade is given by

$$\max(\text{exam}, \frac{\text{exam} + \text{homework}}{2}).$$

Please hand it back during the course, put it in my mailbox at IRMAR, or scan it (with decent quality) and send it by mail to `adrien.laurent@inria.fr`. One homework is expected per person, though the students can work together to tackle the difficult questions. As for academic research, you are encouraged to make drafts and drawings, ask questions, and discuss with the students following the other courses (algebra, geometry,...) to get ideas and advice.

**Crash course in Lie differential geometry:** A Lie group $G$ is a group that is also a differentiable manifold, such that group multiplication and taking inverses are both differentiable. In this assignment, we consider (real) matrix Lie groups, that are, closed subgroups of $\mathrm{GL}_d(\mathbb{R})$. An important property of such manifolds is that their tangent bundle directly relates to their Lie algebra. The Lie algebra $\mathfrak{g}$ associated to $G$ is a vector space defined via the matrix exponential Exp:

$$\mathfrak{g} = \{x \in \mathcal{M}_d(\mathbb{R}), \mathrm{Exp}(x) \in G\}.$$

The Lie algebra $\mathfrak{g}$ coincides with the tangent space $T_eG$ at the identity point $e$ of the group. Moreover, the tangent space at $g \in G$ is given by the matrix multiplication $T_gG = \mathfrak{g}g$.

An homogeneous manifold $\mathcal{M}$ is a manifold on which a Lie group $G$ acts transitively. Equivalently, there exists a subgroup $H$ of $G$ such that $\mathcal{M} = G/H$. Note that Lie groups naturally are homogeneous manifolds. In the matrix case, for all $g \in G$, $p \in \mathcal{M}$, $gp \in \mathcal{M}$. For simplicity, the manifolds considered here are compact and we are not concerned with stability or convergence features.

A vector field $f \in \mathfrak{X}(\mathcal{M})$ is a map satisfying $f(x) \in T_x\mathcal{M}$ for all $x \in \mathcal{M}$. Then, the solution of the differential equation $y'(t) = f(y(t))$, $y(0) = y_0 \in \mathcal{M}$ lies on $\mathcal{M}$. Its solution is the flow associated to $f$. A frame is a collection of vector fields $E_1, \dots, E_m$ such that

$$\mathrm{Span}(E_1(x), \dots, E_m(x)) = T_x\mathcal{M}, \quad x \in \mathcal{M}.$$

Note that every vector field $f \in \mathfrak{X}(G)$ decomposes as

$$f(x) = \sum_{i=1}^{m} f^i(x) E_i(x), \quad x \in \mathcal{M}.$$

In the context of matrix Lie groups, a frame is directly given by $E_i(x) = A_i x$ with $\{A_1, \dots, A_m\}$ a basis of the Lie algebra. It is standard to identify a vector field $E_i(x) = A_i x$ with a differential operator of order one $E_i[\phi](x) = \phi'(x)(A_i x) = \sum_{jk} \partial_j \phi(x)(A_i)_{jk} x_k$.

The aim of this homework is to build high-order integrators on Lie-groups and on homogeneous manifolds.

**Questions**

1. Find the Lie algebra of $G = \mathrm{SO}_d(\mathbb{R})$.

2. Verify that the sphere $\mathcal{S}^2$ is an homogeneous manifold with respect to $\mathrm{SO}_3(\mathbb{R})$ (a drawing/vague explanation is sufficient).

3. Define the rigid body dynamics by

$$f(x) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} x, \quad x_1, x_2, x_3 \in \mathbb{R}.$$

Check that $f$ is a vector field on $\mathcal{S}^2$.

4. Check that $E_i(x) = (0, \dots, 0, 1, 0, \dots, 0)$ (with 1 in position $i$) is a frame on $\mathcal{M} = \mathbb{R}^d$. Given a vector field $f$, an initial point $y_0$, what is the exact solution $y_1$ of the ODE $y' = f_{y_0}(y)$, $y(0) = y_0$ at time $h$ with the frozen vector field $f_{y_0}(x) = \sum f^i(y_0) E_i(x)$? Does it remind you of something?

5. Define a frame $E_1$, $E_2$, $E_3$ on the sphere $\mathcal{S}^2$, and give the exact solution $y_1$ of $y' = f_{y_0}(y)$, $y(0) = y_0$. The map $y_0 \mapsto y_1$ is called the Lie-Euler method.

6. Let $\phi \in \mathcal{C}^\infty(\mathcal{M})$ be a real-valued smooth map and $y$ be the solution of $y' = f(y)$, $y(0) = y_0$. We admit that

$$\phi(y)'(t) = f[\phi](y(t)) = \sum_{i=1}^{m} f^i(y(t)) E_i[\phi](y(t)),$$

where we recall that $f^i \in \mathcal{C}^\infty(\mathcal{M})$, $E_i \in \mathfrak{X}(\mathcal{M})$, and a vector field is analogous to a differential operator of order one. Give the first terms of the Taylor expansion of $\phi(y(h))$ in terms of $f$ and the operation $\rhd$, with $f \rhd g = \sum_{i=1}^m f[g^i] E_i$.

7. Define an algebraic formalism similar to Butcher series to represent the developed Taylor expansion of $\phi(y(h))$ using the Leibniz rule. Give the set of algebraic objects, the elementary differential map $F_f$, and the coefficient map.
*Hint: Be careful. $E_i E_j[\phi] \neq E_j E_i[\phi]$ in general (Schwartz identity does not hold). Think of $E_i$ as $\partial_i$.*

8. For $\alpha \in \mathbb{R}$ and $f \in \mathfrak{X}(\mathcal{M})$, let $\exp(\alpha f)x$ be the solution $y(1)$ of

$$y'(t) = \alpha f(y(t)), \quad y(0) = x.$$

A frozen-flow/Crouch-Grossman integrator is of the form

$$Y_i = \exp(a_{is} h f_{Y_s}) \dots \exp(a_{i1} h f_{Y_1}) y_0$$

$$f_{Y_k} = \sum_{q=1}^{m} f^q(Y_k) E_q$$
$$y_1 = \exp(b_s h f_{Y_s}) \dots \exp(b_1 h f_{Y_1}) y_0.$$

Check that it coincides with a Runge-Kutta method if $\mathcal{M} = \mathbb{R}^d$. Give the first terms of the Taylor expansion of $\phi(y_1)$. Write it in the algebraic formalism and provide the coefficient map.

9. Construct explicit frozen-flow method of order two and three with minimum number of stages.

10. Write an alternate method with only one matrix exponential per stage. This way to define methods is in the spirit of Runge-Kutta-Munthe-Kaas methods.
*Hint: Use BCH.*

11. (Optional) Compare the accuracy of the Lie-Euler method and your frozen-flow methods on the rigid body by implementing them. Take the reference solution as your method with a small time-step for simplicity.

12. Show that $I(y) = y^T y$ is a quadratic invariant for the rigid body. Give an integrator from the course that would preserve $I$. Does it stay on the sphere? Would it work on any manifold?