Multirevolution integrators for differential equations with fast stochastic oscillations

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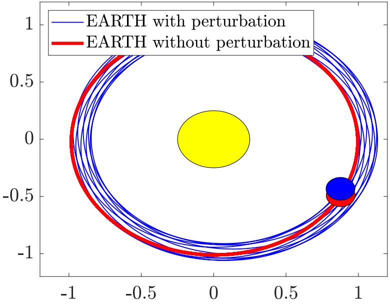
An example with celestial mechanics

Newton model for the motion of earth around the Sun, with $x(t) \in \mathbb{R}^3$ the position of the earth at time t and εP a small perturbation,

$$\frac{d^2x}{dt^2}(t) = \underbrace{-\frac{x(t)}{|x(t)|^3}}_{\text{oscillatory part}} + \underbrace{\varepsilon P\left(x(t), \frac{dx}{dt}(t)\right)}_{\text{small perturbation}}.$$

- If the perturbation $\varepsilon P = 0$, x is T-periodic with $T = 1$ year.
- If $\varepsilon \ll 1$, the motion is pseudo-periodic and x is a perturbation of identity, i.e. $x(t+T) = x(t) + \mathcal{O}(\varepsilon)$.



Example

Take the Solar system with or without Jupiter. After 10 periods (see left Figure), the Earth is almost at the same place as without the perturbation.

The concept of multirevolutions

Multirevolution methods were initially introduced in Melendo, Palacios (1997) and Calvo, Jay, Montijano, Randez (2004) in the context of Astronomy.
The flow $\varphi_{\varepsilon,t}(y)$ of an highly-oscillatory differential equation of the form

$$\frac{dx}{dt}(t) = \underbrace{O(x(t))}_{\text{oscillatory part}} + \underbrace{\varepsilon P\left(x(t)\right)}_{\text{small perturbation}}, \quad x(0) = y$$

is a perturbation of identity over one period T, i.e.

$$\varphi_{\varepsilon,t+T}(y) = \varphi_{\varepsilon,t}(y) + \mathcal{O}(\varepsilon).$$

Goal of multirevolution methods: Integrate the equation after $\mathcal{O}(\varepsilon^{-1})$ periods with cost and accuracy independent of ε.

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Goal of multirevolution methods: Integrate the equation after $\mathcal{O}(\varepsilon^{-1})$ periods with cost and accuracy independent of ε.

Ideas:

- Approximate the flow $\varphi_{\varepsilon,t}(y)$ at the revolution times $t = nT$, $n = 0, 1, 2, \ldots$
- Integrate over $N = \mathcal{O}(\varepsilon^{-1})$ periods at each step using that

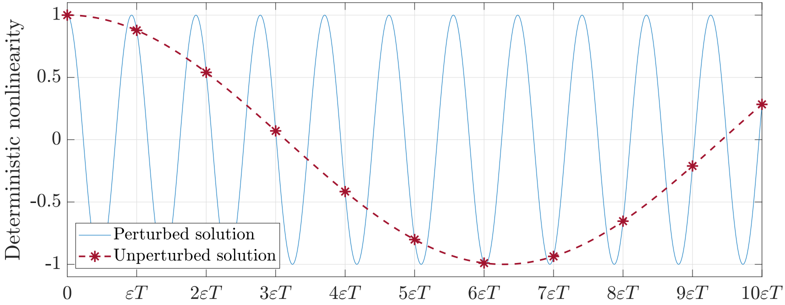
$$\varphi_{\varepsilon,t+NT}(y) = \varphi_{\varepsilon,t}(y) + \mathcal{O}(N\varepsilon).$$

Multirevolution methods for highly oscillatory problems

Previous work using multirevolutions with deterministic oscillatory terms...

- ...on ODEs (see Murua, Sanz-Serna (1999), Calvo, Montijano, Randez (2007) and Chartier, Makazaga, Murua, Vilmart (2014))

$$\frac{dx}{dt}(t) = \frac{1}{\varepsilon}Ax(t) + F(x(t))$$



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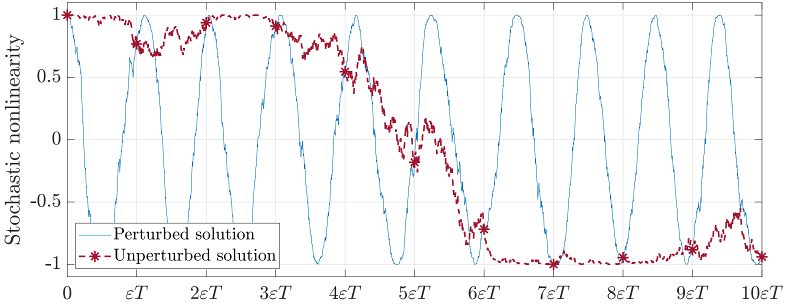
$$\frac{dx}{dt}(t) = \frac{1}{\varepsilon}Ax(t) + F(x(t))$$

- ...on SDEs (see Vilmart (2014))

$$dX(t) = \frac{1}{\varepsilon}AX(t)dt + F(X(t))dt + \sum_k G(X(t))dW_k(t)$$



Differential equations with fast stochastic oscillations

We consider differential equations with fast oscillations driven by a Stratonovich noise

$$dX(t) = \frac{1}{\sqrt{\varepsilon}} AX(t) \circ dW(t) + F(X(t))dt, \; t > 0, \; X(0) = X_0,$$

where $X(t) \in \mathbb{C}^d$, W is a standard one dimensional Brownian motion and

- $e^A = I_d$, that is $\mathrm{Sp}(A) \subset 2i\pi\mathbb{Z}$,
- $\varepsilon \ll 1$,
- F is a smooth nonlinearity.

The above equation can be rewritten with the change of variable $Y(t) = X(\varepsilon t)$ and a rescaled Brownian motion $\widetilde{W}(t) = \frac{1}{\sqrt{\varepsilon}} W(\varepsilon t)$ as

$$dY(t) = \underbrace{AY(t) \circ d\widetilde{W}(t)}_{\text{oscillatory part}} + \underbrace{\varepsilon F(Y(t))dt}_{\text{small perturbation}}, \; t > 0, \; Y(0) = X_0.$$

Related work on long time approximation of SDEs:

A. Laurent and G. Vilmart. Exotic aromatic B-series for the study of long time integrators for a class of ergodic SDEs. *To appear in Math. Comp.*, 2019.

Behaviour in simple cases

Properties of the solution of

$$dY(t) = AY(t) \circ d\widetilde{W}(t) + \varepsilon F(Y(t))dt, \ t > 0, \ Y(0) = X_0$$

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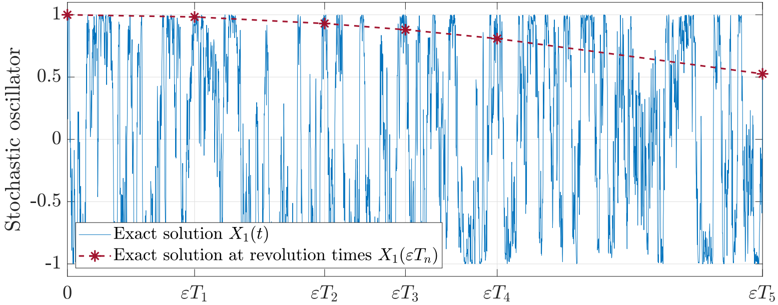
- if $F = 0$, then $Y(t) = e^{A\widetilde{W}(t)}X_0$,
- if $A = 2i\pi$ and $F(y) = iy$, we get a Kubo oscillator and $Y(t) = e^{2i\pi\widetilde{W}(t)}e^{i\varepsilon t}X_0$,



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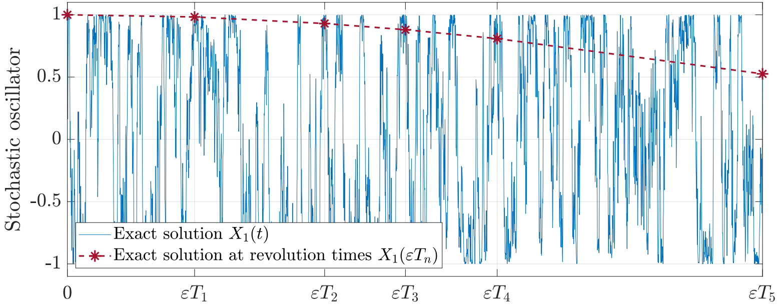
- in the general case, the variation of constants formula yields

$$Y(t) = e^{A\widetilde{W}(t)} X_0 + e^{A\widetilde{W}(t)} \varepsilon \int_0^t e^{-A\widetilde{W}(s)} F(Y(s)) ds.$$



The nonlinear Schrödinger equation in fiber optics

The equation governing the amplitude of the pulse going through an optical fiber with varying dispersion coefficient m is the following nonlinear Schrödinger equation

$$\partial_t u(t,x) = i\nu m(t)\partial_x^2 u(t,x) + \nu^2 F(u(t,x)), \quad u(t=0,x) = u_0(x).$$

where F is typically of the form $F(u) = i\left|u\right|^2 u$.
When ν tends to 0, $u^\nu(t,x) = u(t/\nu^2,x)$ converges to the solution of the nonlinear Schrödinger equation with white noise dispersion,

$$\partial_t u(t,x) = i\partial_x^2 u(t,x) \circ \frac{dW}{dt}(t) + F(u(t,x)), \quad u(t=0,x) = u_0(x).$$

In the context of SPDEs, see Marty (2006), Agrawal (2007, 2008), De Bouard, Debussche (2010).

Highly-oscillatory SDEs in fiber optics

If the initial data is small, we derive the following more general SPDE

$$\begin{cases} du(t) &= \frac{i}{\sqrt{\varepsilon}}\Delta u(t) \circ dW(t) + F(u(t))dt, \quad x \in \mathbb{T}^d, \quad t > 0, \\ u(0) &= u_0, \quad x \in \mathbb{T}^d. \end{cases}$$

A spectral discretization with K modes yields the following real differential equation with fast stochastic oscillations

$$dX(t) = \frac{1}{\sqrt{\varepsilon}} AX(t) \circ dW(t) + F(X(t))dt, \ t > 0, \ X(0) = X_0,$$

with $A = \mathrm{Diag}(-2i\pi k^2, |k| \leqslant K)$ and $e^A = I_d$.

Related work on numerical integrators for $\varepsilon = 1$: exponential integrators (Cohen (2012), Cohen, Dujardin (2017), Erdoğan, Lord (2018)), split-step method (Marty (2006)) or Crank-Nicholson scheme (Belaouar, De Bouard, Debussche (2015)).

Aim of the talk

- Integrate numerically the following highly oscillatory SDE when $\varepsilon \ll 1$,

$$dX(t) = \frac{1}{\sqrt{\varepsilon}} AX(t) \circ dW(t) + F(X(t))dt, \ t > 0, \ X(0) = X_0.$$

- Study the asymptotic regime $\varepsilon \to 0$.
- Derive and analyse a new numerical method of weak order 2 based on the idea of multirevolutions and an invariant preserving modification.

Method A (Explicit integrator of weak order two in $H = N\varepsilon$)

$Y_0 = X_0$

for $m \geqslant 0$ **do**

$$Y_{m+1} = Y_m + H \sum_{k\in\mathbb{Z}} c_k^0(Y_m)\widehat{\alpha}_k^N + H^2 \sum_{p,k\in\mathbb{Z}} c_p^1(Y_m)(c_k^0(Y_m)\widehat{\beta}_{p,k}^N)$$

end for

Contents

Deriving a local expansion in ε: variation of constants

Change of variable $t \to \frac{t}{\varepsilon}$ with a rescaled Brownian motion $\widetilde{W}(t) = \frac{1}{\sqrt{\varepsilon}} W(\varepsilon t)$:

$$dY(t) = AY(t) \circ d\widetilde{W}(t) + \varepsilon F(Y(t))dt,\ t > 0,\ Y(0) = y.$$

Notation

We denote $\varphi_{\varepsilon,t}(y) = Y(t)$ the flow of the equation above.

Goal: approximate $\varphi_{\varepsilon,t}(y)$ at time with size $\mathcal{O}(\varepsilon^{-1})$ with a cost independent of ε.

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Properties on $\varphi_{\varepsilon,t}(y)$:

- if $F = 0$, then $\varphi_{\varepsilon,t}(y) = e^{A\widetilde{W}(t)} y$,
- in the general case, the variation of constants formula yields

$$\varphi_{\varepsilon,t}(y) = e^{A\widetilde{W}(t)} y + \varepsilon \int_0^t e^{A(\widetilde{W}(t) - \widetilde{W}(s))} F(\varphi_{\varepsilon,s}(y)) ds.$$

Stroboscopic approximation for highly oscillatory ODEs

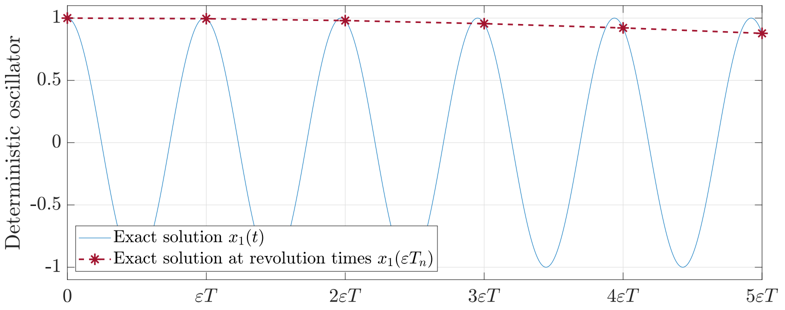
Example

Linear oscillator: $\frac{dx}{dt}(t) = \frac{2i\pi}{\varepsilon}x(t) + ix(t)$

Exact solution: $x(t) = e^{2i\pi\varepsilon^{-1}t}e^{it}x_0$

Revolution times: $T_n = n$

Exact solution evaluated at revolution times: $x(\varepsilon T_n) = \underbrace{e^{2i\pi T_n}}_{=I_d} e^{i\varepsilon T_n}x_0$



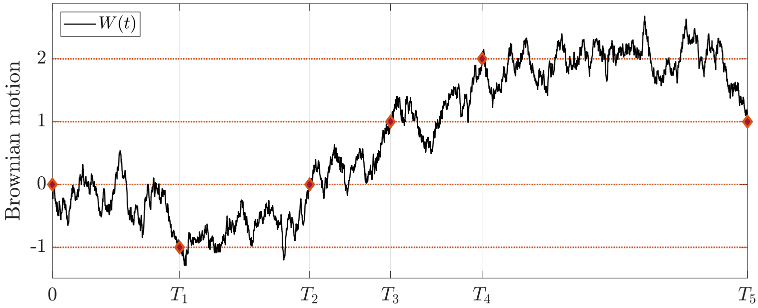
Revolution times

Issue: $e^{A\widetilde{W}(t)}$ is not periodic in contrast to e^{At}.

We define the revolution times of $\widetilde{W}(t)$ as the random variables

$$T_0 = 0,$$
$$T_{n+1} = \inf\left\{t > T_n, \left|\widetilde{W}(t) - \widetilde{W}(T_n)\right| \geqslant 1\right\}, \ n = 0, 1, \dots$$

Then as $e^A = I_d$, we find $e^{A\widetilde{W}(T_n)} = I_d$.



Stroboscopic approximation for the Kubo oscillator

Example

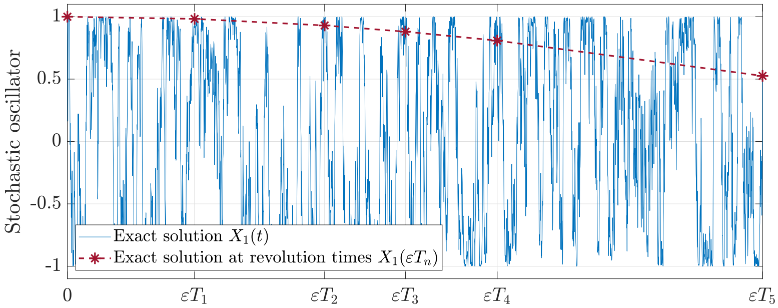
Kubo oscillator: $dX(t) = \frac{2i\pi}{\sqrt{\varepsilon}} X(t) \circ dW(t) + iX(t)dt$

Exact solution: $X(t) = e^{2i\pi\varepsilon^{-1/2}W(t)} e^{it} X_0$

Revolution times: $T_0 = 0, \; T_{n+1} = \inf\left\{t > T_n, \left|\widetilde{W}(t) - \widetilde{W}(T_n)\right| \geqslant 1\right\}$

Exact solution evaluated at revolution times:

$$X(\varepsilon T_n) = e^{2i\pi\varepsilon^{-1/2}W(\varepsilon T_n)} e^{i\varepsilon T_n} X_0 = e^{2i\pi \widetilde{W}(T_n)} e^{i\varepsilon T_n} X_0 = e^{i\varepsilon T_n} X_0$$



Deriving a local expansion in ε: iterative expansions

Variation of constants formula:

$$\varphi_{\varepsilon,t}(y) = e^{AW(t)}y + \varepsilon \int_0^t e^{A(W(t)-W(s))}F(\varphi_{\varepsilon,s}(y))ds.$$

We formally derive local expansions of the exact solution at any order.

Order 0:

$$\varphi_{\varepsilon,t}(y) = e^{AW(t)}y + \mathcal{O}(\varepsilon t)$$

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Order 2:

$$\begin{aligned}\varphi_{\varepsilon,t}(y) = & e^{AW(t)}y + \varepsilon e^{AW(t)} \int_0^t e^{-AW(s)} F(e^{AW(s)}y) ds \\ & + \varepsilon^2 e^{AW(t)} \int_0^t e^{-AW(s)} F'(e^{AW(s)}y) \left(e^{AW(s)} \int_0^s e^{-AW(r)} F(e^{AW(r)}y) dr \right) ds \\ & + \mathcal{O}((\varepsilon t)^3) = \psi_{\varepsilon,t}(y) + \mathcal{O}((\varepsilon t)^3)\end{aligned}$$

Deriving a local expansion in ε: approximation at T_N

We now consider $t = T_N$ (revolution time), the exact flow $\varphi_{\varepsilon,T_N}(y)$ simplifies to the following perturbation of identity:

$$\begin{aligned}\varphi_{\varepsilon,T_N}(y) = {} & y+\varepsilon \int_0^{T_N} e^{-AW(s)} F(e^{AW(s)}y)ds \\ & +\varepsilon^2 \int_0^{T_N} e^{-AW(s)} F'(e^{AW(s)}y) \left(e^{AW(s)} \int_0^s e^{-AW(r)} F(e^{AW(r)}y)dr \right) ds \\ & + \mathcal{O}((\varepsilon T_N)^3)\end{aligned}$$

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Proposition

$\psi_{\varepsilon,T_N}(y)$ is a strong order 2 approximation of $\varphi_{\varepsilon,T_N}(y)$, that is

$$\mathbb{E}\left[\left| \varphi_{\varepsilon,T_N}(y) - \psi_{\varepsilon,T_N}(y) \right|^2 \right]^{1/2} \leqslant C(1+|y|^K) \underbrace{(\varepsilon N)^3}_{H^3}.$$

Deriving a local expansion in ε: approximation at T_N

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Proof.

The Gronwall lemma yields an estimate of the form

$$|\varphi_{\varepsilon,t}(y)-\psi_{\varepsilon,t}(y)| \leqslant C(1+|y|^K)e^{C\varepsilon t}(\varepsilon t)^3.$$

Thus when evaluated at T_N, one gets

$$\mathbb{E}\left[\left|\varphi_{\varepsilon,T_N}(y)-\psi_{\varepsilon,T_N}(y)\right|^2\right]^{1/2} \leqslant C(1+|y|^K)\mathbb{E}[e^{C\varepsilon T_N}(\varepsilon T_N)^3].$$

The existence of the Laplace transform $\mathbb{E}[e^{sT_1}]$ of T_1 for all s small enough implies $\mathbb{E}[e^{C\varepsilon T_N}(\varepsilon T_N)^3] \leqslant C(\varepsilon N)^3$ for all $\varepsilon \leqslant \varepsilon_0$. Hence the result. □

Construction of the methods

We obtain the following order 2 approximation of $\varphi_{\varepsilon,T_N}(y)$:

$$\psi_{\varepsilon,T_N}(y) = y + \varepsilon \int_0^{T_N} e^{-AW(s)} F(e^{AW(s)} y) ds$$
$$+ \varepsilon^2 \int_0^{T_N} e^{-AW(s)} F'(e^{AW(s)} y) \left(e^{AW(s)} \int_0^s e^{-AW(r)} F(e^{AW(r)} y) dr \right) ds$$

Issue: The above long time integrals involve F and F'.

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$$\psi_{\varepsilon,T_N}(y) = y + (\varepsilon N) \sum_k c_k^0(y) \frac{1}{N} \int_0^{T_N} e^{2i\pi kW(s)} ds$$
$$+ (\varepsilon N)^2 \sum_{p,k} c_p^1(y) \left(c_k^0(y) \frac{1}{N^2} \int_0^{T_N} e^{2i\pi pW(s)} \int_0^s e^{2i\pi kW(r)} drds \right)$$

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$$\psi_{\varepsilon,T_N}(y) = y + (\varepsilon N) \sum_k c_k^0(y) \alpha_k^N + (\varepsilon N)^2 \sum_{p,k} c_p^1(y) \left(c_k^0(y) \beta_{p,k}^N \right)$$

with

$$\alpha_k^N = \frac{1}{N} \int_0^{T_N} e^{2i\pi k W(s)} ds$$
$$\beta_{p,k}^N = \frac{1}{N^2} \int_0^{T_N} e^{2i\pi p W(s)} \int_0^s e^{2i\pi k W(r)} dr ds.$$

Construction of the methods

We deduce the following numerical scheme of order 2 for approximating the exact solution $\varphi_{\varepsilon,T_{Nm}}(X_0)$, $m = 0, 1, \ldots$.

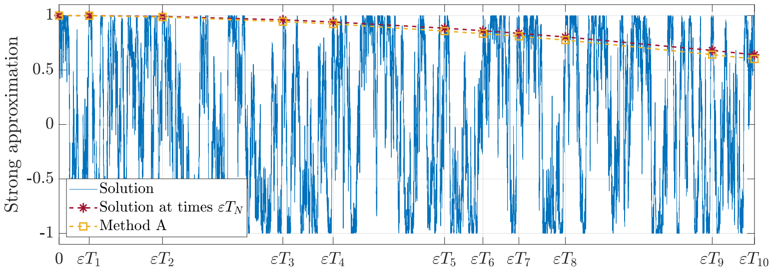
Method A (Explicit integrator of strong order two in $H = N\varepsilon$)

$Y_0 = X_0$

for $m \geqslant 0$ **do**

$$Y_{m+1} = Y_m + H \sum_{k\in\mathbb{Z}} c_k^0(Y_m)\alpha_k^N + H^2 \sum_{p,k\in\mathbb{Z}} c_p^1(Y_m)(c_k^0(Y_m)\beta_{p,k}^N)$$

end for



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for $m \geqslant 0$ **do**

$$Y_{m+1} = Y_m + H \sum_{k\in\mathbb{Z}} c_k^0(Y_m)\alpha_k^N + H^2 \sum_{p,k\in\mathbb{Z}} c_p^1(Y_m)(c_k^0(Y_m)\beta_{p,k}^N)$$

end for

Remark: This method can be generalized to create strong numerical schemes of any order under proper regularity assumptions on the nonlinearity F.

Issue: a standard approximation of the integrals α_k^N and $\beta_{p,k}^N$ has a cost in $\mathcal{O}(\varepsilon^{-1})$.

Contents

Weak order 2 approximation of the weak integrals

We obtained the following strong/weak approximation of order 2:

$$\psi_{\varepsilon,T_N}(y) = y + H\sum_k c_k^0(y)\alpha_k^N + H^2\sum_{p,k} c_p^1(y)(c_k^0(y)\beta_{p,k}^N).$$

However, computing exactly α_k^N and $\beta_{p,k}^N$ has a cost in $\mathcal{O}(\varepsilon^{-1})$. We introduce

$$\widehat{\psi}_{\varepsilon,N}(y) = y + H\sum_k c_k^0(y)\widehat{\alpha}_k^N + H^2\sum_{p,k} c_p^1(y)(c_k^0(y)\widehat{\beta}_{p,k}^N),$$

where we replaced α_k^N and $\beta_{p,k}^N$ with cheap discrete approximations with same first moments $\widehat{\alpha}_k^N$ and $\widehat{\beta}_{p,k}^N$ (see Milstein, Tretyakov (2004)), that is

$$\mathbb{E}[\widehat{\alpha}_k^N] = \mathbb{E}[\alpha_k^N],\ \mathbb{E}[\widehat{\beta}_{p,k}^N] = \mathbb{E}[\beta_{p,k}^N],$$

$$\mathbb{E}[\mathrm{Re}(\widehat{\alpha}_{k_1}^N)\,\mathrm{Re}(\widehat{\alpha}_{k_2}^N)] = \mathbb{E}[\mathrm{Re}(\alpha_{k_1}^N)\,\mathrm{Re}(\alpha_{k_2}^N)],$$

$$\mathbb{E}[\mathrm{Re}(\widehat{\alpha}_{k_1}^N)\,\mathrm{Im}(\widehat{\alpha}_{k_2}^N)] = \mathbb{E}[\mathrm{Re}(\alpha_{k_1}^N)\,\mathrm{Im}(\alpha_{k_2}^N)],$$

$$\mathbb{E}[\mathrm{Im}(\widehat{\alpha}_{k_1}^N)\,\mathrm{Im}(\widehat{\alpha}_{k_2}^N)] = \mathbb{E}[\mathrm{Im}(\alpha_{k_1}^N)\,\mathrm{Im}(\alpha_{k_2}^N)].$$

First and second moments of α_k^N and $\beta_{p,k}^N$

Proposition

The following random variables

$$\begin{aligned} \alpha_k^N &= \frac{1}{N}\int_0^{T_N} e^{2i\pi k W(s)} ds \\ \beta_{p,k}^N &= \frac{1}{N^2}\int_0^{T_N} e^{2i\pi p W(s)} \int_0^s e^{2i\pi k W(r)} drds \end{aligned}$$

satisfy

$$\begin{aligned} \mathbb{E}[\alpha_k^N] &= \delta_k = \begin{cases} 1 \text{ if } k = 0 \\ 0 \text{ else} \end{cases} \\ \mathbb{E}[\alpha_p^N \alpha_k^N] &= \begin{cases} 1 + \frac{2}{3N} \text{ if } p = k = 0 \\ \frac{1}{\pi^2 p^2 N} \text{ if } p + k = 0,\ p, k \neq 0 \\ 0 \text{ else} \end{cases} \\ \mathbb{E}[\beta_{p,k}^N] &= \begin{cases} \frac{1}{2} + \frac{1}{3N} \text{ if } p = k = 0 \\ \frac{1}{2\pi^2 k^2 N} \text{ if } p = 0,\ k \neq 0 \\ \frac{-1}{2\pi^2 p^2 N} \text{ if } p \neq 0,\ k = 0 \\ \frac{1}{2\pi^2 p^2 N} \text{ if } p + k = 0,\ p, k \neq 0 \\ 0 \text{ else} \end{cases} \end{aligned}$$

Proof and an application

Idea of proof.

The Itô formula applied to $e^{2i\pi kW(s)}$ for $k \neq 0$ yields

$$\alpha_k^N = \frac{i}{\pi kN} \int_0^{T_N} e^{2i\pi kW(s)} dW(s).$$

The Doob theorem allows to conclude as $t \mapsto \int_0^t e^{2i\pi kW(s)} dW(s)$ is a martingale. □

Remark (Stochastic Fourier series)

Let f be a L^2 function on $]0,1[$ extended on $\mathbb{R}$ by 1-periodicity, whose Fourier coefficients are denoted as $(c_k)_{k\in\mathbb{Z}}$, then

$$\mathbb{E}\left[\int_0^{T_1} f(W(s))ds\right] = c_0 = \int_0^1 f(\theta)d\theta \text{ and } \mathbb{E}\left[\int_0^{T_1} |f(W(s))|^2 ds\right] = \sum_k |c_k|^2$$

Euler method and asymptotic regime $\varepsilon \to 0$

We have the following approximation of order 1:

$$\psi_{\varepsilon,N}(y) = y + H \sum_k c_k^0(y) \alpha_k^N.$$

If we replace α_k^N by $\widehat{\alpha}_k^N = \mathbb{E}[\alpha_k^N] = \delta_k$, we get the Euler method

$$y_{M+1} = y_M + H c_0^0(y_M).$$

It has weak order 1 in $H = N\varepsilon$ and cost independent of N and ε.

Theorem (Asymptotic model)

Under regularity assumptions on F, the exact solution $\varphi_{\varepsilon,T_{T/\varepsilon}}(X_0) = Y(T_{T/\varepsilon})$ converges weakly as $\varepsilon \to 0$ to the solution at time T of the deterministic ODE

$$\frac{dy_t}{dt} = \langle g^0 \rangle (y_t) \left(= \int_0^1 e^{-A\theta} F(e^{A\theta} y_t) d\theta \right), \ y_0 = X_0.$$

Remark: This asymptotic model is the same one as for deterministic oscillations.

Euler method and asymptotic regime $\varepsilon \to 0$

We have the following approximation of order 1:

$$\psi_{\varepsilon,N}(y) = y + H \sum_k c_k^0(y) \alpha_k^N.$$

If we replace α_k^N by $\widehat{\alpha}_k^N = \mathbb{E}[\alpha_k^N] = \delta_k$, we get the Euler method

$$y_{M+1} = y_M + H c_0^0(y_M).$$

It has weak order 1 in $H = N\varepsilon$ and cost independent of N and ε.

Example

On the Kubo oscillator $dY = AY \circ dW + i\varepsilon Y dt$, this amounts to do the approximation

$$\begin{aligned} e^{i\varepsilon T_N} &\approx 1 + iH \frac{T_N}{N} + \mathcal{O}(H^2) \text{ (strong approximation)} \\ &\approx 1 + iH \underbrace{\mathbb{E}\left[\frac{T_N}{N}\right]}_{=1} + \mathcal{O}(H^2) \text{ (weak approximation)} \end{aligned}$$

New robust order 2 method

Method A (Explicit integrator of weak order two in $H = N\varepsilon$)

$Y_0 = X_0$
for $m \geqslant 0$ **do**

$$Y_{m+1} = Y_m + H \sum_{k\in\mathbb{Z}} c_k^0(Y_m)\widehat{\alpha}_k^N + H^2 \sum_{p,k\in\mathbb{Z}} c_p^1(Y_m)(c_k^0(Y_m)\widehat{\beta}_{p,k}^N)$$

end for

Theorem (Weak convergence of order 2)

Under regularity assumption on F, Method A is a weak order 2 integrator for approximating $\varphi_{\varepsilon,T_{Nm}}(X_0) \approx Y_m$ with $m = 0, 1, \dots,$ that is

$$|\mathbb{E}[\phi(\varphi_{\varepsilon,T_{Nm}}(X_0))] - \mathbb{E}[\phi(Y_m)]| \leqslant CH^2(1 + \mathbb{E}[|X_0|^K]).$$

Remarks: The cost is linear in the number of Fourier modes.
The method can be adapted to approximate the solution at a deterministic time T with the same cost and accuracy.

New geometric robust order 2 method

Geometric modification based on the implicit middle point method for preserving quadratic invariants, where $\widetilde{\beta}_{p,k}^N = \beta_{p,k}^N - \frac{\alpha_p^N \alpha_k^N}{2}$. For example, for the Schrödinger equation, if $F(y) = i\,|y|^{2\sigma}\, y$, the L^2 norm $Q(y) = y^T y$ is preserved.

Method B (Geometric integrator of weak order two in $H = N\varepsilon$)

$Y_0 = X_0$

for $m \geqslant 0$ **do**

$$Y_{m+1} = Y_m + H \sum_{k\in\mathbb{Z}} c_k^0 \left(\frac{Y_m + Y_{m+1}}{2}\right) \widehat{\alpha}_k^N + H^2 \sum_{p,k\in\mathbb{Z}} c_p^1 \left(\frac{Y_m + Y_{m+1}}{2}\right) \left(c_k^0 \left(\frac{Y_m + Y_{m+1}}{2}\right)\right) \widehat{\widetilde{\beta}}_{p,k}^N$$

end for

Theorem

Under regularity assumption on F, Method B is a weak order 2 algorithm for approximating $\varphi_{\varepsilon,T_{Nm}}(X_0)$ with $m = 0, 1, \dots$ and preserves quadratic invariants.

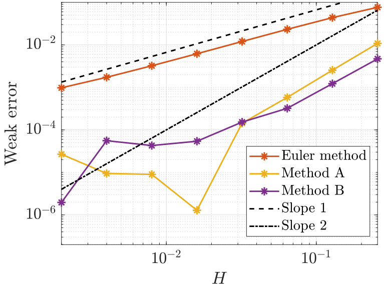
Contents

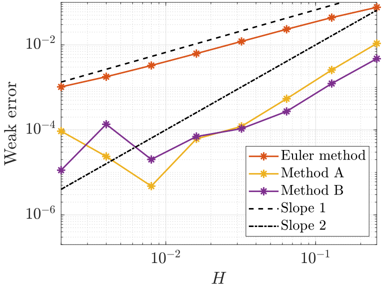
Weak order of convergence

We solve numerically the following equation for the linear $F(u) = iu$ (left) and the non-linear $F(u) = i(1 + \Re(u)^3 + \Im(u)^5)u$ (right)

$$dX(t) = \frac{2i\pi}{\sqrt{\varepsilon}} X(t) \circ dW(t) + F(X(t))dt, \ X(0) = 1.$$

We plot on a logarithmic scale an estimate of the weak error ($\sim 10^6$ trajectories) with both methods for approximating X at time $T = 10^{-3} T_{2^8}$ where $\mathbb{E}[T] = 0.256$. We observe a convergence of order 2, which corroborates the weak order 2 convergence theorems of Method A and B.





Highly oscillatory NLS with white noise dispersion

We apply our algorithms to a spatial discretization (with 2^7 modes) of the SPDE

$$du = \frac{i}{\sqrt{\varepsilon}} \Delta u \circ dW + i\,|u|^{2\sigma} u dt, \;\; u_0(x) = \exp(-3x^4 + x^2), \;\; x \in [-\pi, \pi].$$

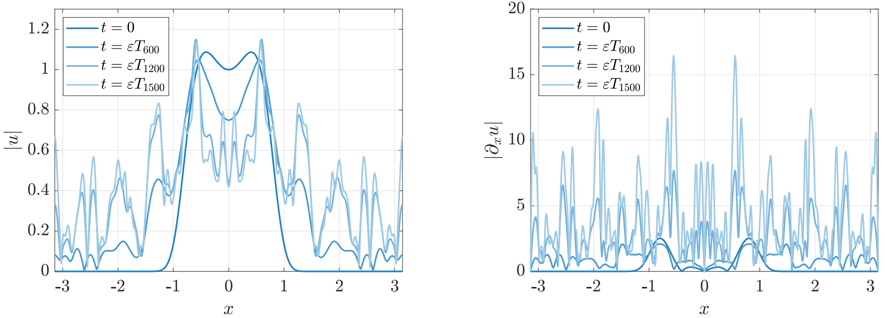


Figure: Approximation of $|u|$ and $|\partial_x u|$ for $\sigma = 4$ and $\varepsilon = 10^{-2}$.

Behaviour of L^2 and H^1 norms

Properties of the equation $du = \frac{i}{\sqrt{\varepsilon}}\Delta u \circ dW + F(u)dt$ with $F(u) = i\,|u|^{2\sigma}\,u$:

- The L^2 norm of the exact solution is constant.
- Conjecture of Belaouar, De Bouard, Debussche (2015): the H^1 norm of the exact solution explodes in finite time for $\sigma \geqslant 4$ (critical exponent in deterministic case $\sigma \geqslant 2$).

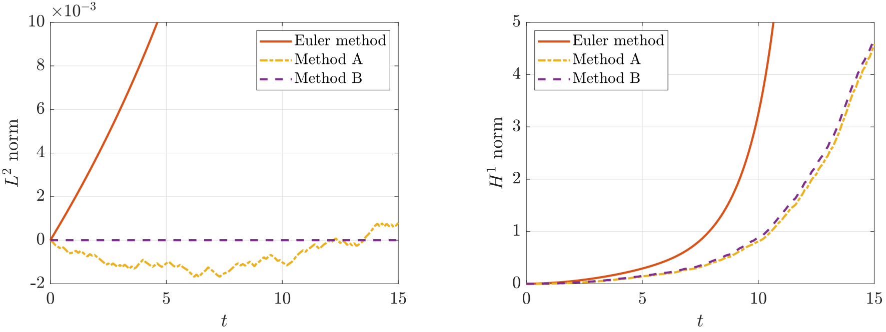


Figure: Evolution of the quantity $\big|\|\psi_{\varepsilon,t}(u_0)\| - \|u_0\|\big|$ with the discrete L^2 and H^1 norms for $\sigma = 4$, $\varepsilon = 10^{-2}$ and $u_0(x) = \exp(-3x^4 + x^2)$.

Summary

- We give a method to obtain asymptotic expansions in ε of the flow of

$$dX(t) = \frac{1}{\sqrt{\varepsilon}} AX(t) \circ dW(t) + F(X(t))dt, \ t > 0, \ X(0) = X_0.$$

- We build a method of weak order two based on the idea of multirevolutions with computational cost and accuracy both independent of the stiffness of the oscillations ε.
- We propose a geometric modification that conserves exactly quadratic invariants.
- There exists an asymptotic model ($\varepsilon \to 0$) and it is the same one as for deterministic oscillations.
- Possible further research on uniformly accurate schemes.

Main reference of this talk:

A. Laurent and G. Vilmart. Multirevolution integrators for differential equations with fast stochastic oscillations. *Submitted*, arXiv:1902.01716, 2019.