Multirevolution integrators for differential equations with fast stochastic oscillations

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The concept of multirevolutions

Multirevolution methods were initially introduced in Melendo, Palacios (1997) and Calvo, Jay, Montijano, Randez (2004) in the context of Astronomy.
Consider an highly-oscillatory differential equation of the form

$$\frac{dx}{dt}(t) = \underbrace{O(x(t))}_{\text{oscillatory part}} + \underbrace{\varepsilon P(x(t))}_{\text{small perturbation}}, \quad x(0) = y.$$

- **Issue:** Standard integrators usually have a stepsize restriction $h \leqslant C\varepsilon$ for stability/accuracy.
- **Goal of multirevolution methods:** Integrate the equation after $\mathcal{O}(\varepsilon^{-1})$ periods with cost and accuracy independent of ε.
- **Ideas:**
  The flow $\varphi_{\varepsilon,t}(y) = x(t)$ of is a perturbation of identity over one period T, i.e.

$$\varphi_{\varepsilon,t+T}(y) = \varphi_{\varepsilon,t}(y) + \mathcal{O}(\varepsilon).$$

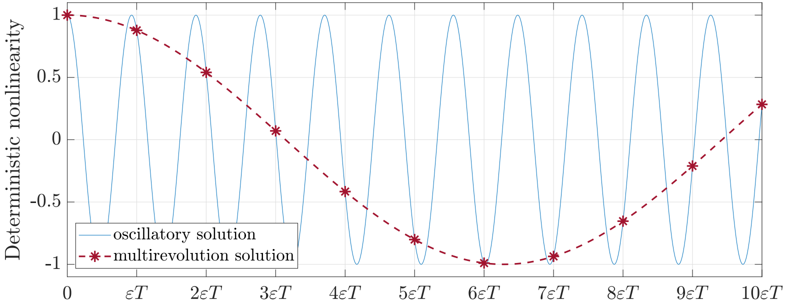
  Approximate the flow $\varphi_{\varepsilon,t}(y)$ at the revolution times $t = nT$, $n = 0, 1, 2, \ldots$

Multirevolution methods for highly oscillatory problems

Previous work using multirevolutions with deterministic oscillatory terms...

- ...on ODEs (see Murua, Sanz-Serna (1999), Calvo, Montijano, Randez (2007) and Chartier, Makazaga, Murua, Vilmart (2014))

$$\frac{dx}{dt}(t) = \frac{1}{\varepsilon}Ax(t) + F(x(t))$$



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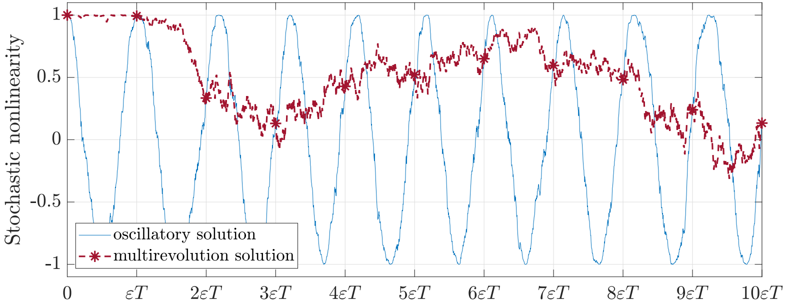
$$\frac{dx}{dt}(t) = \frac{1}{\varepsilon}Ax(t) + F(x(t))$$

- ...on SDEs (see Vilmart (2014))

$$dX(t) = \frac{1}{\varepsilon}AX(t)dt + F(X(t))dt + \sum_k G(X(t))dW_k(t)$$



Differential equations with fast stochastic oscillations

We consider differential equations with fast oscillations driven by a Stratonovich noise

$$dX(t) = \frac{1}{\sqrt{\varepsilon}} AX(t) \circ dW(t) + F(X(t))dt, \ t > 0, \ X(0) = X_0,$$

where $X(t) \in \mathbb{C}^d$, W is a standard one dimensional Brownian motion and

- $e^A = I_d$, that is $\mathrm{Sp}(A) \subset 2i\pi\mathbb{Z}$,
- $\varepsilon \ll 1$,
- F is a smooth nonlinearity.

The above equation can be rewritten with the change of variable $Y(t) = X(\varepsilon t)$ and a rescaled Brownian motion $\widetilde{W}(t) = \frac{1}{\sqrt{\varepsilon}} W(\varepsilon t)$ as

$$dY(t) = \underbrace{AY(t) \circ d\widetilde{W}(t)}_{\text{oscillatory part}} + \underbrace{\varepsilon F(Y(t))dt}_{\text{small perturbation}}, \ t > 0, \ Y(0) = X_0.$$

Related work on long time approximation of SDEs:

A. Laurent and G. Vilmart. Exotic aromatic B-series for the study of long time integrators for a class of ergodic SDEs. *To appear in Math. Comp.*, 2019.

Behaviour in simple cases

Properties of the solution of

$$dY(t) = AY(t) \circ d\widetilde{W}(t) + \varepsilon F(Y(t))dt, \;\; t > 0, \;\; Y(0) = X_0$$

Behaviour in simple cases

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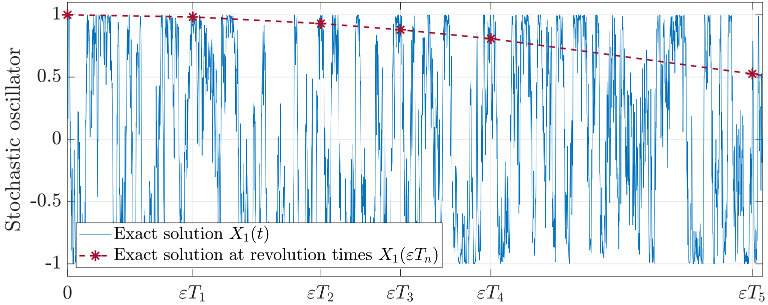
Behaviour in simple cases

Properties of the solution of

$$dY(t) = AY(t) \circ d\widetilde{W}(t) + \varepsilon F(Y(t))dt, \; t > 0, \; Y(0) = X_0$$

- if $F = 0$, then $Y(t) = e^{A\widetilde{W}(t)} X_0$,
- if $A = 2i\pi$ and $F(y) = iy$, we get a Kubo oscillator and $Y(t) = e^{2i\pi\widetilde{W}(t)} e^{i\varepsilon t} X_0$,



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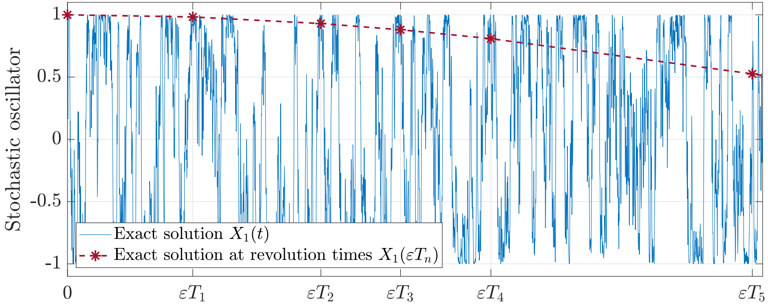
- if $F = 0$, then $Y(t) = e^{A\widetilde{W}(t)} X_0$,
- if $A = 2i\pi$ and $F(y) = iy$, we get a Kubo oscillator and $Y(t) = e^{2i\pi\widetilde{W}(t)} e^{i\varepsilon t} X_0$,
- in the general case, the variation of constants formula yields

$$Y(t) = e^{A\widetilde{W}(t)} X_0 + e^{A\widetilde{W}(t)} \varepsilon \int_0^t e^{-A\widetilde{W}(s)} F(Y(s)) ds.$$



Highly-oscillatory SDEs in fiber optics

The equation governing the amplitude of the pulse going through an optical fiber with a varying dispersion coefficient is the following nonlinear Schrödinger equation with white noise dispersion (see Marty (2006), Agrawal (2007, 2008), De Bouard, Debussche (2010))

$$\begin{cases} du(t) &= \frac{i}{\sqrt{\varepsilon}}\Delta u(t)\circ dW(t) + F(u(t))dt, \quad x\in\mathbb{T}^d, \quad t>0, \\ u(0) &= u_0, \quad x\in\mathbb{T}^d. \end{cases}$$

A spectral discretization with K modes yields the following differential equation with fast stochastic oscillations

$$dX(t) = \frac{1}{\sqrt{\varepsilon}}AX(t)\circ dW(t) + F(X(t))dt,\ t>0,\ X(0)=X_0,$$

with $A = \mathrm{Diag}(-2i\pi k^2, |k|\leqslant K)$ and $e^A = I_d$.

Related work on numerical integrators for $\varepsilon = 1$: exponential integrators (Cohen (2012), Cohen, Dujardin (2017), Erdoğan, Lord (2018)), split-step method (Marty (2006)) or Crank-Nicholson scheme (Belaouar, De Bouard, Debussche (2015)).

Contents

Stroboscopic approximation for highly oscillatory ODEs

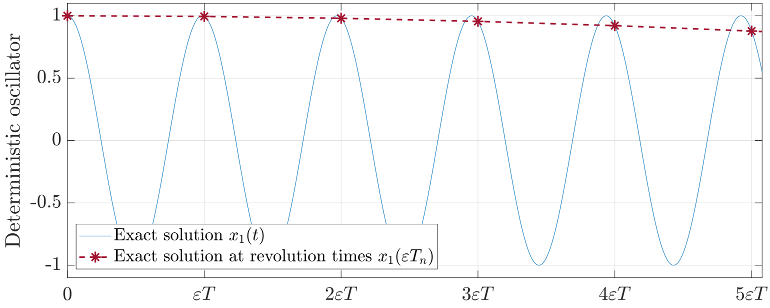
Example

Linear oscillator: $\frac{dx}{dt}(t) = \frac{2i\pi}{\varepsilon}x(t) + ix(t)$

Exact solution: $x(t) = e^{2i\pi\varepsilon^{-1}t}e^{it}x_0$

Revolution times: $T_n = n$

Exact solution evaluated at revolution times: $x(\varepsilon T_n) = \underbrace{e^{2i\pi T_n}}_{=I_d} e^{i\varepsilon T_n}x_0$



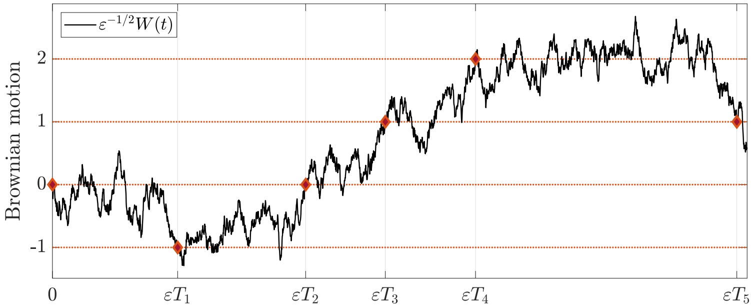
Revolution times

Issue: $e^{A\widetilde{W}(t)}$ is not periodic in contrast to e^{At}.

We define the revolution times of $\widetilde{W}(t)$ as the random variables

$$T_0 = 0,$$
$$T_{n+1} = \inf\left\{t > T_n, \left|\widetilde{W}(t) - \widetilde{W}(T_n)\right| \geqslant 1\right\}, \ n = 0, 1, \ldots$$

Then as $e^A = I_d$, we find $e^{A\widetilde{W}(T_n)} = I_d$.



Stroboscopic approximation for the Kubo oscillator

Example

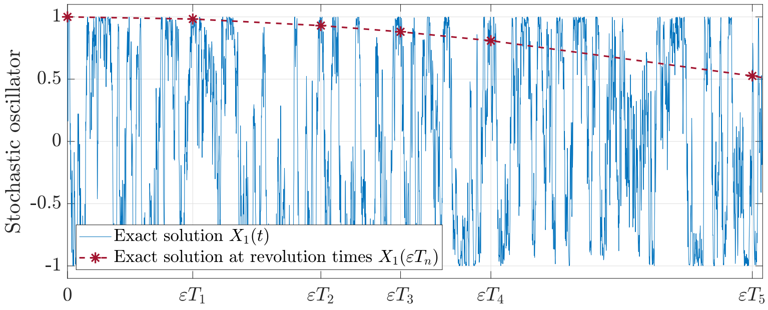
Kubo oscillator: $dX(t) = \frac{2i\pi}{\sqrt{\varepsilon}} X(t) \circ dW(t) + iX(t)dt$

Exact solution: $X(t) = e^{2i\pi\varepsilon^{-1/2}W(t)} e^{it} X_0$

Revolution times: $T_0 = 0, \; T_{n+1} = \inf \left\{ t > T_n, \left| \widetilde{W}(t) - \widetilde{W}(T_n) \right| \geqslant 1 \right\}$

Exact solution evaluated at revolution times:

$$X(\varepsilon T_n) = e^{2i\pi\varepsilon^{-1/2}W(\varepsilon T_n)} e^{i\varepsilon T_n} X_0 = e^{2i\pi \widetilde{W}(T_n)} e^{i\varepsilon T_n} X_0 = e^{i\varepsilon T_n} X_0$$



Deriving a local expansion in ε: iterative expansions

Variation of constants formula:

$$\varphi_{\varepsilon,t}(y) = e^{AW(t)}y + \varepsilon \int_0^t e^{A(W(t)-W(s))}F(\varphi_{\varepsilon,s}(y))ds.$$

We formally derive local expansions of the exact solution at any order.

Order 0:

$$\varphi_{\varepsilon,t}(y) = e^{AW(t)}y + \mathcal{O}(\varepsilon t)$$

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$$\varphi_{\varepsilon,t}(y) = e^{AW(t)}y + \varepsilon e^{AW(t)} \int_0^t e^{-AW(s)} F(e^{AW(s)}y)ds + \mathcal{O}((\varepsilon t)^2)$$

Deriving a local expansion in ε: iterative expansions

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Order 2:

$$\begin{aligned}\varphi_{\varepsilon,t}(y) = {} & e^{AW(t)}y + \varepsilon e^{AW(t)} \int_0^t e^{-AW(s)} F(e^{AW(s)}y)ds \\ & + \varepsilon^2 e^{AW(t)} \int_0^t e^{-AW(s)} F'(e^{AW(s)}y) \left(e^{AW(s)} \int_0^s e^{-AW(r)} F(e^{AW(r)}y)dr \right) ds \\ & + \mathcal{O}((\varepsilon t)^3) = \psi_{\varepsilon,t}(y) + \mathcal{O}((\varepsilon t)^3)\end{aligned}$$

Deriving a local expansion in ε: approximation at T_N

We now consider $t = T_N$ (revolution time), the exact flow $\varphi_{\varepsilon,T_N}(y)$ simplifies to the following perturbation of identity:

$$\begin{aligned}\varphi_{\varepsilon,T_N}(y) = {} & y+\varepsilon\int_0^{T_N} e^{-AW(s)}F(e^{AW(s)}y)ds \\ & +\varepsilon^2\int_0^{T_N} e^{-AW(s)}F'(e^{AW(s)}y)\left(e^{AW(s)}\int_0^s e^{-AW(r)}F(e^{AW(r)}y)dr\right)ds \\ & + \mathcal{O}((\varepsilon T_N)^3)\end{aligned}$$

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$$+ \mathcal{O}((\varepsilon T_N)^3) = \psi_{\varepsilon,T_N}(y) + \underbrace{\mathcal{O}((\varepsilon T_N)^3)}_{???}$$

Proposition (L., Vilmart)

$\psi_{\varepsilon,T_N}(y)$ is a strong order 2 approximation of $\varphi_{\varepsilon,T_N}(y)$, that is

$$\mathbb{E}\left[\left| \varphi_{\varepsilon,T_N}(y) - \psi_{\varepsilon,T_N}(y) \right|^2 \right]^{1/2} \leqslant C(1 + |y|^K) \underbrace{(\varepsilon N)^3}_{=H^3} .$$

Construction of the methods

We obtain the following order 2 approximation of $\varphi_{\varepsilon,T_N}(y)$:

$$\psi_{\varepsilon,T_N}(y) = y + \varepsilon \int_0^{T_N} e^{-AW(s)} F(e^{AW(s)} y) ds$$
$$+ \varepsilon^2 \int_0^{T_N} e^{-AW(s)} F'(e^{AW(s)} y) \left(e^{AW(s)} \int_0^s e^{-AW(r)} F(e^{AW(r)} y) dr \right) ds$$

Issue: The above long time integrals involve F and F'.

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If $c_k^0(y)$ is the k^{th} Fourier coefficient of $g_\theta^0(y) = e^{-A\theta} F(e^{A\theta} y)$, $c_p^1(y)$ the p^{th} Fourier coefficient of $g_\theta^1(y) = e^{-A\theta} F'(e^{A\theta} y)(e^{A\theta} \cdot)$, then

$$\psi_{\varepsilon,T_N}(y) = y + (\varepsilon N) \sum_k c_k^0(y) \frac{1}{N} \int_0^{T_N} e^{2i\pi k W(s)} ds + (\varepsilon N)^2 \sum_{p,k} c_p^1(y)(c_k^0(y)) \frac{1}{N^2} \int_0^{T_N} e^{2i\pi p W(s)} \int_0^s e^{2i\pi k W(r)} drds$$

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$$\psi_{\varepsilon,T_N}(y) = y + (\varepsilon N) \sum_k c_k^0(y) \alpha_k^N + (\varepsilon N)^2 \sum_{p,k} c_p^1(y)(c_k^0(y)) \beta_{p,k}^N$$

with

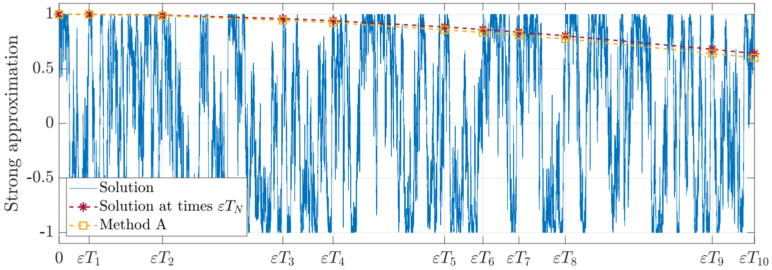
$$\alpha_k^N = \frac{1}{N} \int_0^{T_N} e^{2i\pi k W(s)} ds$$
$$\beta_{p,k}^N = \frac{1}{N^2} \int_0^{T_N} e^{2i\pi p W(s)} \int_0^s e^{2i\pi k W(r)} drds.$$

Construction of the methods

We deduce the following numerical scheme of order 2 for approximating the exact solution $\varphi_{\varepsilon,T_{Nm}}(X_0) = X(\varepsilon T_{Nm})$, $m = 0, 1, \dots$.

Method A (Explicit integrator of strong order two in $H = N\varepsilon$)

$$Y_{m+1} = Y_m + H \sum_{k\in\mathbb{Z}} c_k^0(Y_m)\alpha_k^N + H^2 \sum_{p,k\in\mathbb{Z}} c_p^1(Y_m)(c_k^0(Y_m)\beta_{p,k}^N)$$



Issue: a standard approximation of the integrals $\alpha_k^N = \frac{1}{N}\int_0^{T_N} e^{2i\pi kW(s)} ds$ and $\beta_{p,k}^N = \frac{1}{N^2}\int_0^{T_N} e^{2i\pi pW(s)} \int_0^s e^{2i\pi kW(r)} drds$ has a cost $\mathcal{O}(\varepsilon^{-1})$.

Contents

Weak order 2 approximation of the weak integrals

We obtained the following strong/weak approximation of order 2:

$$\psi_{\varepsilon,T_N}(y) = y + H\sum_k c_k^0(y)\alpha_k^N + H^2\sum_{p,k} c_p^1(y)(c_k^0(y))\beta_{p,k}^N.$$

However, computing exactly α_k^N and $\beta_{p,k}^N$ has a cost in $\mathcal{O}(\varepsilon^{-1})$. We introduce

$$\widehat{\psi}_{\varepsilon,N}(y) = y + H\sum_k c_k^0(y)\widehat{\alpha}_k^N + H^2\sum_{p,k} c_p^1(y)(c_k^0(y))\widehat{\beta}_{p,k}^N,$$

where we replaced α_k^N and $\beta_{p,k}^N$ with cheap discrete approximations with same first moments $\widehat{\alpha}_k^N$ and $\widehat{\beta}_{p,k}^N$ (see Milstein, Tretyakov (2004)), that is

$$\mathbb{E}[\widehat{\alpha}_k^N] = \mathbb{E}[\alpha_k^N],\ \mathbb{E}[\widehat{\beta}_{p,k}^N] = \mathbb{E}[\beta_{p,k}^N],$$

$$\mathbb{E}[\widehat{\alpha}_{k_1}^N\widehat{\alpha}_{k_2}^N] = \mathbb{E}[\alpha_{k_1}^N\alpha_{k_2}^N].$$

First and second moments of α_k^N and $\beta_{p,k}^N$

Proposition

The following random variables

$$\begin{aligned} \alpha_k^N &= \frac{1}{N}\int_0^{T_N} e^{2i\pi k W(s)} ds \\ \beta_{p,k}^N &= \frac{1}{N^2}\int_0^{T_N} e^{2i\pi p W(s)} \int_0^s e^{2i\pi k W(r)} dr ds \end{aligned}$$

satisfy

$$\mathbb{E}[\alpha_k^N] = \delta_k = \begin{cases} 1 \textit{ if } k = 0 \\ 0 \textit{ else} \end{cases}$$

$$\mathbb{E}[\alpha_p^N \alpha_k^N] = \begin{cases} 1 + \frac{2}{3N} \textit{ if } p = k = 0 \\ \frac{1}{\pi^2 p^2 N} \textit{ if } p + k = 0,\ p,k \neq 0 \\ 0 \textit{ else} \end{cases}$$

$$\mathbb{E}[\beta_{p,k}^N] = \begin{cases} \frac{1}{2} + \frac{1}{3N} \textit{ if } p = k = 0 \\ \frac{1}{2\pi^2 k^2 N} \textit{ if } p = 0,\ k \neq 0 \\ \frac{-1}{2\pi^2 p^2 N} \textit{ if } p \neq 0,\ k = 0 \\ \frac{1}{2\pi^2 p^2 N} \textit{ if } p + k = 0,\ p,k \neq 0 \\ 0 \textit{ else} \end{cases}$$

Euler method and asymptotic regime $\varepsilon \to 0$

We have the following approximation of order 1:

$$\psi_{\varepsilon,N}(y) = y + H \sum_k c_k^0(y) \alpha_k^N.$$

If we replace α_k^N by $\widehat{\alpha}_k^N = \mathbb{E}[\alpha_k^N] = \delta_k$, we get the Euler method

$$y_{M+1} = y_M + H c_0^0(y_M).$$

It has weak order 1 in $H = N\varepsilon$ and cost independent of N and ε.

Theorem (L., Vilmart)

Under regularity assumptions on F, the exact solution $\varphi_{\varepsilon, T_{T/\varepsilon}}(X_0) = Y(T_{T/\varepsilon})$ converges weakly as $\varepsilon \to 0$ to the solution at time T of the deterministic ODE

$$\frac{dy_t}{dt} = \langle g^0 \rangle (y_t) \left(= \int_0^1 e^{-A\theta} F(e^{A\theta} y_t) d\theta \right), \; y_0 = X_0.$$

Remark: This asymptotic model is the same one as for deterministic oscillations.

New robust order 2 method

Method A (Explicit integrator of weak order two in $H = N\varepsilon$)

$$Y_{m+1} = Y_m + H \sum_{k\in\mathbb{Z}} c_k^0(Y_m)\widehat{\alpha}_k^N + H^2 \sum_{p,k\in\mathbb{Z}} c_p^1(Y_m)(c_k^0(Y_m)\widehat{\beta}_{p,k}^N)$$

Theorem (L., Vilmart)

Under regularity assumptions on F, Method A is a weak order 2 integrator for approximating $\varphi_{\varepsilon,T_{Nm}}(X_0) = X(\varepsilon T_{Nm}) \approx Y_m$ with $m = 0, 1, \dots,$ that is

$$|\mathbb{E}[\phi(\varphi_{\varepsilon,T_{Nm}}(X_0))] - \mathbb{E}[\phi(Y_m)]| \leqslant CH^2(1 + \mathbb{E}[|X_0|^K]).$$

Remarks:

- The cost is linear in the number of Fourier modes (indexed by k).
- The method can be adapted to approximate the solution at a deterministic time T with the same cost and accuracy.

New geometric robust order 2 method

Geometric modification based on the implicit middle point method for preserving quadratic invariants, where $\widetilde{\beta}^N_{p,k} = \beta^N_{p,k} - \frac{\alpha^N_p \alpha^N_k}{2}$. For example, for the Schrödinger equation, if $F(y) = i\,|y|^{2\sigma}\, y$, the L^2 norm $Q(y) = y^T y$ is preserved.

Method B (Geometric integrator of weak order two in $H = N\varepsilon$)

$$Y_{m+1} = Y_m + H \sum_{k\in\mathbb{Z}} c^0_k \left(\frac{Y_m + Y_{m+1}}{2} \right) \widehat{\alpha}^N_k + H^2 \sum_{p,k\in\mathbb{Z}} c^1_p \left(\frac{Y_m + Y_{m+1}}{2} \right) \left(c^0_k \left(\frac{Y_m + Y_{m+1}}{2} \right) \right) \widehat{\widetilde{\beta}}^N_{p,k}$$

Theorem (L., Vilmart)

Under regularity assumptions on F, Method B is a weak order 2 integrator for approximating $\varphi_{\varepsilon,T_{Nm}}(X_0) = X(\varepsilon T_{Nm})$ with $m = 0, 1, \ldots$ and preserves quadratic invariants.

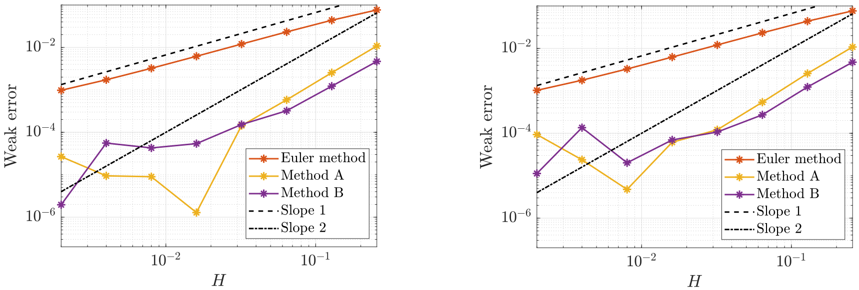
Contents

Weak order of convergence

$$dX(t) = \frac{2i\pi}{\sqrt{\varepsilon}} X(t) \circ dW(t) + F(X(t))dt, \ X(0) = 1$$

We plot on a logarithmic scale an estimate of the weak error ($\sim 10^6$ trajectories) with both methods for approximating X at time $T = 10^{-3} T_{2^8}$ where $\mathbb{E}[T] = 0.256$. We observe a convergence of order 2, which corroborates the weak order 2 convergence theorems of Method A and B.

$F(u) = iu$ $\qquad$ $F(u) = i(1 + \Re(u)^3 + \Im(u)^5)u$



Highly oscillatory NLS with white noise dispersion

We apply our algorithms to a spatial discretization (with 2^7 modes) of the SPDE

$$du = \frac{i}{\sqrt{\varepsilon}}\Delta u \circ dW + i\,|u|^{2\sigma}\, udt,\ \ u_0(x) = \exp(-3x^4 + x^2),\ \ x \in [-\pi, \pi].$$

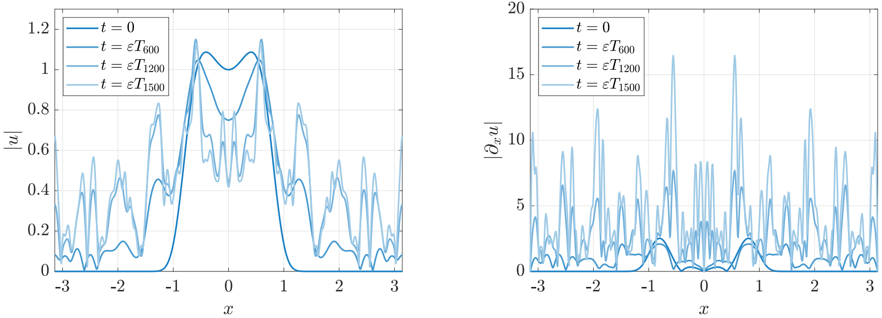


Figure: Approximation of $|u|$ and $|\partial_x u|$ for $\sigma = 4$ and $\varepsilon = 10^{-2}$.

Behaviour of L^2 and H^1 norms

Properties of the equation $du = \frac{i}{\sqrt{\varepsilon}}\Delta u \circ dW + F(u)dt$ with $F(u) = i\,|u|^{2\sigma}\,u$:

- The L^2 norm of the exact solution is constant.
- Conjecture of Belaouar, De Bouard, Debussche (2015) for $\varepsilon = 1$: the H^1 norm of the exact solution explodes in finite time for $\sigma \geqslant 4$ (critical exponent in the deterministic case $\sigma \geqslant 2$).

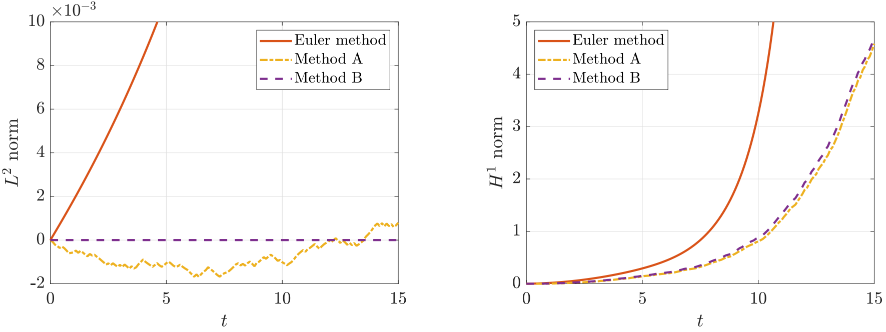


Figure: Evolution of the quantity $|\|\psi_{\varepsilon,t}(u_0)\| - \|u_0\||$ with the discrete L^2 and H^1 norms for $\sigma = 4$, $\varepsilon = 10^{-2}$ and $u_0(x) = \exp(-3x^4 + x^2)$.

Summary

- We give a method to obtain asymptotic expansions in ε of the flow of

$$dX(t) = \frac{1}{\sqrt{\varepsilon}} AX(t) \circ dW(t) + F(X(t))dt, \ t > 0, \ X(0) = X_0.$$

- We build a method of weak order two based on the idea of multirevolutions with computational cost and accuracy both independent of the stiffness of the oscillations ε.
- We propose a geometric modification that conserves exactly quadratic invariants.
- There exists an asymptotic model ($\varepsilon \to 0$) and it is the same one as for deterministic oscillations.
- Possible further research on uniformly accurate schemes.

Preprint available:

A. Laurent and G. Vilmart. Multirevolution integrators for differential equations with fast stochastic oscillations. *Submitted*, arXiv:1902.01716, 2019.