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Introduction

ODE : $\begin{cases} \dot{x}(t) = b(x(t)) \\ x(0) = x_0 \end{cases}$

$b : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field
$X : [0,\infty) \to \mathbb{R}^n$ is the time-dependent trajectory, solution of the system
$X(t)$ is the state of the system at time $t \geq 0$.

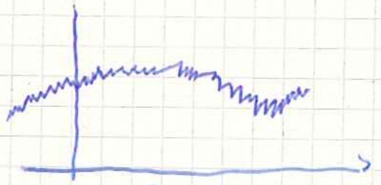
Illustration :

The solution is also smooth (for b smooth enough)

$$x(t) = x_0 + \int_0^t b(x(t))\, dt$$

Model for smooth trajectory. For example
- solar system
- A ball you launched
- etc.

There is other class of problems where the solution seems to follow an ODE, but with random perturbation



This can't be modelised by an ODE because the trajectory is not smooth and there is no randomness in an ODE

We need to add this randomness somehow to the system:

$$\begin{cases} \dot{X}(t) = b(X(t)) + \underbrace{B(X(t))\,\xi(t)}_{\text{random part}} \\ X(0) = x_0 \end{cases}$$

$B : \mathbb{R}^n \to M^{n\times m}$, $\xi : \mathbb{R} \to \mathbb{R}^m$ is the m-dimensional white-noise

We need to define what this white noise is.

In the case

$\begin{cases} \dot{X} = \xi(t) \\ X(0) = 0 \end{cases}$, the solution is $X(t) = W(t)$, the n dimensional brownian motion.

That is $\xi(\cdot) = \dot{W}(t)$

SDE $\Rightarrow \dfrac{dX}{dt} = b(X(t))\, dt + B(X(t))\dfrac{dW}{dt}$

$\Rightarrow$ (SDE) $\begin{cases} dX(t) = b(X(t))\,dt + B(X(t))\,dW(t) \\ X(0) = x_0 \end{cases}$

dX, dW are called stochastic differentials

$X(t)$ solves the SDE if $X(t) = x_0 + \underbrace{\int_0^{t} b(X(s))\,dt}_{\text{well defined}} + \underbrace{\int_0^t B(X(s))\,dW}_{\text{what does it mean?}}$ $t \geq 0$

for $n = m = 1$.

<u>Itô's chain rule</u>: Let X be such that

$$dX = b(X)\,dt + C(X)\,dW.$$

Let $u : \mathbb{R} \to \mathbb{R}$ a smooth function. Let $Y(t) = u(X(t))$

What is dY ?

In the deterministic case: $dY = u' dX = u' b\,dt + u' C\,dW$

$\leadsto$ This is not true however in our case because $dW \approx (dt)^{1/2}$

$$Y + dY = u(X + dX) = Y + u' dX + \frac{1}{2} u'' (dX)^2 + \dots$$

$$\begin{aligned} dY &= u' dX + \frac{1}{2} u'' (dX)^2 + \dots \\ &= u'(b\,dt + C\,dW) + \frac{1}{2} u'' (b\,dt + C\,dW)^2 + \dots \\ &= u' b\,dt + u' C\,dW + \frac{1}{2} u'' b\,dt^2 + u'' b\,dt\,C\,dW + \frac{1}{2} u'' C^2 \underbrace{dW^2}_{= dt} + \dots \\ &= \left(u' b + \frac{1}{2} u'' C^2\right) dt + u' C\,dW + O(dt^{3/2}) \end{aligned}$$

$$\boxed{du(X) = \left(u' b + \frac{C^2}{2} u''\right) dt + C u'\,dW}$$

<u>Ex</u>: $\begin{cases} dY = Y\,dW \\ Y(0) = 1 \end{cases}$ Then $Y(t) = e^{W(t) - \frac{t}{2}}$

Let $X = -\frac{t}{2} + W(t)$ That is $dX = -\frac{1}{2} dt + dW$

$\Rightarrow b = -\frac{1}{2}$, $C = 1$

Let $u(X) = e^X$

Then $dY = du(X) = \Big(\underbrace{e^X \left(-\tfrac{1}{2}\right) + \tfrac{1}{2} e^X}_{= 0}\Big) dt + e^X dW = Y(t)\,dW$

<u>Ex</u> Let $S(t)$ be the price of a stock at time $t > 0$.

The relative change of price $\frac{dS}{S} = \mu\,dt + \sigma\,dW$, where $\mu > 0$ is the drift and σ the volatility

volatility : degree of variation of stock price

drift : The expected return

$\Rightarrow \begin{cases} dS = \mu S\,dt + \sigma S\,dW \\ S(0) = S_0 \end{cases}$, that is S_0 is the starting price

Then $S(t) = S_0 e^{\sigma W(t) + (\mu - \frac{\sigma^2}{2})t}$

Let $X(t) = \sigma W(t) + (\mu - \frac{\sigma^2}{2})t$. Then $dX = (\mu - \frac{\sigma^2}{2})dt + \sigma dW$

Let $u(X) = e^X$ $\Rightarrow b = \mu - \frac{\sigma^2}{2}$, $c = \sigma$

Then $dS = du(X) = \left(e^X(\mu - \frac{\sigma^2}{2}) + \frac{\sigma^2}{2} e^X\right)dt + \sigma e^X dW$

$= \mu e^X dt + \sigma e^X dW = \mu S dt + \sigma S dW$

Plan:
1) Recall of ODE and proba.
2) Brownian motion
3) Stochastic integral + Itô's formula
4) SDE
5) Numerical analysis for SDE

ODE : $\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$ | Let E be a Banach space, $D \subset E$ open and convex. Let I be an interval of $\mathbb{R}$.

$f : I \times D \to E$, $y : I \to D$ is the solution of the system.
We want to show existence and unicity of the solution for the system under proper hypothesis. $\hat{y} \in C^1(I, D)$

Def: $f : I \times D \to E$ is said to be globally lipschitz for x if $\exists L > 0$ st.
$\forall x_1, x_2 \in D$, $\forall t \in I$

$$\|f(t, x_1) - f(t, x_2)\|_E \leq L \|x_1 - x_2\|_E$$

Thm Let $X \subset E$ a closed subset of E. Let $F : X \to X$ be a contraction (ie $L < 1$), then $\exists!\, y \in X$ st $F(y) = y$.

Recall that we have $y(t) = y_0 + \int_{t_0}^{t} f(t, y(t))\, dt$

Thm Assume $D = E$. Assume $f \in C(I \times D, E)$ be globally lipschitz for x. Then $\forall y_0 \in D$ $\exists!\, y \in C^1(I, D)$ st $\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$

Proof: Let us assume I closed and bounded. Let L be the lipschitz constant of f.
Let $\mathcal{E} = (C(I, E), \|\cdot\|_{\mathcal{E}})$ with $\|y\|_{\mathcal{E}} = \max_{t \in I} e^{-2L|t - t_0|} \|y(t)\|_E$
$\|\cdot\|_{\mathcal{E}}$ is equivalent to $\|\cdot\|_\infty$ since, if we denote $C = \min_{t \in I} e^{-2L|t - t_0|} > 0$ (since I closed and bounded)

$$C \|y\|_\infty = C \max_{t \in I} \|y(t)\|_E \leq \max_{t \in I} e^{-2L|t - t_0|} \|y(t)\|_E \leq \max_{t \in I} \|y(t)\|_E = \|y\|_\infty$$

$\mathcal{E}$ is complet since I is compact.

Let us define $T : \mathcal{E} \to \mathcal{E}$ with $\forall t \in I$
$(Ty)(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$
It is clear that $Ty \in \mathcal{E}$ since f is continuous. Assume $t > t_0$

$$\begin{aligned}
\|Ty_1(t) - Ty_2(t)\|_E &\leq \left\| \int_{t_0}^{t} f(s, y_1(s)) - f(s, y_2(s))\, ds \right\|_E \\
&\leq \int_{t_0}^{t} \|f(s, y_1(s)) - f(s, y_2(s))\|_E\, ds \\
&\leq \int_{t_0}^{t} L \|y_1(s) - y_2(s)\|_E\, ds \\
&= \int_{t_0}^{t} L e^{2L|s - t_0|} e^{-2L|s - t_0|} \|y_1(s) - y_2(s)\|_E\, ds
\end{aligned}$$

$$\leq \int_{t_0}^{t} L e^{2L|s-t_0|} \cdot \max_{t\in I} \cdot e^{-2L|t-t_0|} \|y_1 - y_2\|_\Sigma \, ds$$

$$= \int_{t_0}^{t} L e^{2L|s-t_0|} \|y_1 - y_2\|_\Sigma \, ds = \|y_1 - y_2\|_\Sigma L \underbrace{\int_{t_0}^{t} e^{2L|s-t_0|} ds}_{} \leq$$

$$\leq \frac{1}{2} e^{2L|t-t_0|} \|y_1 - y_2\|_\Sigma$$

$$= \begin{cases} \left.\dfrac{e^{2L(s-t_0)}}{2L}\right|_{t_0}^{t} & \text{if } t > t_0 \\ \left.\dfrac{e^{2L(t_0-s)}}{-2L}\right|_{t}^{t_0} & \text{if } t < t_0 \end{cases} = \begin{cases} \dfrac{e^{2L(t-t_0)}}{2L} - \dfrac{1}{2L} < \dfrac{e^{2L|t-t_0|}}{2L} \\ -\dfrac{1}{2L} + \dfrac{e^{2L(t_0-t)}}{2L} \leq \dfrac{e^{2L|t-t_0|}}{2L} \end{cases}$$

$\Rightarrow \|Ty_1 - Ty_2\|_\Sigma \leq \frac{1}{2} \|y_1 - y_2\|_\Sigma$

$\Rightarrow T$ is a contraction from Σ to Σ

$\Rightarrow \exists!\, y \in \Sigma$ st $Ty = y$

Assume I either not closed or not bounded.

Then $I = \bigcup_{n\in\mathbb{N}} I_n$ with $I_n \subset I_{n+1}$, I_n bounded and closed

Let y_n be the solution on I_n. By unicity $y_{n+1}|_{I_n} = y_n$

Then y is define st $y = y_n$ on I_n

This imply unicity and existence of the solution. □

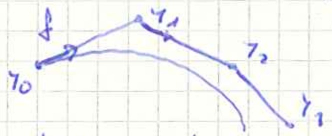
<u>Numerical methods for ODE:</u>

We want to approximate the solution of $\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$ i.e. $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$

Let $T > 0$, $N \in \mathbb{N}$, $h = \frac{T}{N}$, $t_n = n \cdot h$

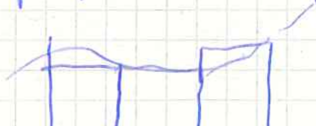
<u>Euler explicit:</u> $y_{n+1} = y_n + h f(t_n, y_n)$

Illustration:



This is ~~equivalent~~ can bee seen as an approximation of $\int_{t_n}^{t_{n+1}} f(s, y(s))\, ds \approx h f(t_n, y(t_n))$

<u>Euler implicit:</u> $y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})$:



That is $\int_{t_n}^{t_{n+1}} f(s, y(s))\, ds \approx h f(t_{n+1}, y(t_{n+1}))$

Def We say a method has local order p if $\|y(t_1) - y_1\| = O(h^{p+1})$ $\forall h$ suff. small

<u>Example</u>

Euler explicit: $y_1 - (y(t_0) + h y'(t_0) + O(h^2)) = y_1 - (\underbrace{y_0 + h f(t_0, y_0)}_{y_1} + O(h^2))$

$= O(h^2)$

Probability :

Definitions : $(\Omega, U, \mathbb{P})$: probability space

Ω is the universe, domain. $\omega \in \Omega$ is a sample point.

U is a σ-algebra and represent the information. $A \in U$ is an event.

$\mathbb{P}$ is the probability, a measure with respect to (Ω, U) ~~and ([0,1], $\mathcal{B}$)~~

~~of~~ σ-algebra is a collection U of subsets of Ω st

- $\emptyset, \Omega \in U$
- $A \in U \Rightarrow A^c \in U$
- $A_1, A_2, \dots \in U \Rightarrow \bigcup_{k=1}^{\infty} A_k, \bigcap_{k=1}^{\infty} A_k \in U$

•) $\mathbb{P} : U \to [0,1]$ is such that : i) $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$

ii) $\mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} \mathbb{P}(A_k)$

$\mathbb{P}\left(\dot{\bigcup}_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(A_k)$

We say $A \perp\!\!\!\perp B$, A, B indep if $P(A \cap B) = P(A)P(B)$

$X_i : \Omega \to \mathbb{R}$ r.v. $\mathbb{P}(X_1 \in B_1, \dots, X_n \in B_n) = \mathbb{P}(X_1 \in B_1) \cdots \mathbb{P}(X_n \in B_n)$ $\forall B_k \in \mathcal{B}_k$

•) A property which holds ~~for all~~ except for A with $\mathbb{P}(A) = 0$ hold almost surely

•) The Borel σ-algebra of $\mathbb{R}^n$ is the smallest σ-alg $\mathcal{B}$ that contains all open sets of $\mathbb{R}^n$.

•) $X : \Omega \to \mathbb{R}^n$ is an n-dimensional random variable (r.v.) if $\forall B \in \mathcal{B}$ $X^{-1}(B) \in U$. $X : \Omega \to \mathbb{R}^n$, $= (X_1, \dots, X_n)$ is an n-dimensional random vector X_i is a r.v.

• $U(X) := \{X^{-1}(B) \mid B \in \mathcal{B}\}$ is the σ-alg generated by X (lemma)

$U(X)$ contains all the relevant information about X.

Example : $\Omega = \{⚀, ⚁, ⚂, ⚅, ⚄, ⚃\} =: \{1, \dots, 6\}$

$U = \{\text{all subsets of } \Omega\}$

$\mathbb{P}(A) = \frac{|A|}{6} \quad \forall A \in U$

$$X(\omega) = \begin{cases} 0 & \text{if } \omega \le 3 \\ 1 & \text{if } \omega > 3 \end{cases} \quad : U(X) = \{\emptyset, \{1,2,3\}, \{4,5,6\}, \Omega\}$$

$$X(\omega) = \{ n \text{ if } \omega = n \qquad U(X) = U$$

Def: i) The distribution function of X is $F_X : \mathbb{R}^n \to [0,1]$ st

$F_X(x) := P(X \le x) \quad \forall x \in \mathbb{R}^n$

(ii) $X_1, \dots, X_m : \Omega \to \mathbb{R}^n$ r.v., their joint distribution function is $F_{X_1 \dots X_m} : (\mathbb{R}^n)^m \to [0,1]$ st $F_{X_1, \dots, X_m}(x_1, \dots, x_m) := \mathbb{P}(X_1 \le x_1, \dots, X_m \le x_m)$)

- A **density function** for X is a nonnegative, integrable function $f : \mathbb{R}^n \to \mathbb{R}$ s.t.

$$F(x) = F(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(y_1, \ldots, y_n)\, dy_n \ldots dy_1$$

- The expected value of $X : \Omega \to \mathbb{R}$ is:

$$\mathbb{E}(X) = \int_\Omega X \, d\mathbb{P} = \int_{\mathbb{R}^n} x f(x)\, dx \quad , \quad \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) \text{ if } X \perp\!\!\!\perp Y$$

(and $\mathbb{E}(X) < \infty$, $\mathbb{E}(Y) < \infty$)

$X : \Omega \to \mathbb{R}^n$ $\quad \mathbb{E}(X) = \begin{pmatrix} \mathbb{E}(X_1) \\ \mathbb{E}(X_n) \end{pmatrix}$

- The variance of X is

$$V(X) = \int_\Omega [X - \mathbb{E}(X)]^2 \, d\mathbb{P} = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \int_{\mathbb{R}^n} |x - m|^2 f(x)\, dx$$

$V(X+Y) = V(X) + V(Y)$

- The covariance of X, Y is $Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ | Remark $Cov(X,X) = Var(X)$

$Cov(X,Y) = 0$ if $X \perp\!\!\!\perp Y$

- $(X_i)_{i=1..n}$ iid if $X_i \perp\!\!\!\perp X_j$ and X_i same law

Exemple: $X : \Omega \to \mathbb{R}$ has Gaussian law with mean m and variance σ^2 if the density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

We write $X \sim N(m, \sigma^2)$. $\quad$ / X_i, X_j iid, $X_i \sim N(m_i, \sigma_i^2)$

$X_i + X_j \sim N(m_i + m_j, \sigma_i^2 + \sigma_j^2)$

$\alpha X_i \sim N(\alpha m_i, \alpha^2 \sigma_i^2)$

$$\mathbb{E}(X) = \int_{\mathbb{R}} x f_X(x)\, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} (m + x - m) e^{-\frac{(x-m)^2}{2\sigma^2}} dx =$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \underbrace{\left[-\sigma^2 e^{-\frac{(x-m)^2}{2\sigma^2}} \right]_{-\infty}^{\infty}}_{=0} + m \underbrace{\int_{\mathbb{R}} f_X(x)\, dx}_{=1} = m$$

- $X : \Omega \to \mathbb{R}^n$ has gaussian law if $a \cdot X$ has gaussian law $\forall a \in \mathbb{R}^n$

This implies $f(x) = \dfrac{1}{((2\pi)^n \det \Gamma)^{1/2}} e^{-\frac{1}{2}(x-m)\Gamma^{-1}(x-m)}$

with $(c_{ij}) = Cov(X_i, X_j)$

Thm (central limit) $X_1, \ldots, X_n, \ldots$ iid. $\mathbb{E}(X_i) = m$, $V(X_i) = \sigma^2 > 0$

$S_n =: X_1 + \ldots + X_n$

Then $\forall a, b \quad -\infty < a < b < \infty$, $\quad \frac{\sqrt{n}}{\sigma} \cdot \left[\frac{X_1 + \ldots + X_n}{n} - m \right] \xrightarrow{\mathcal{L}} N(0,1)$

i.e. $\frac{X_1 + \ldots + X_n}{n} \approx N(m, \frac{\sigma^2}{n})$

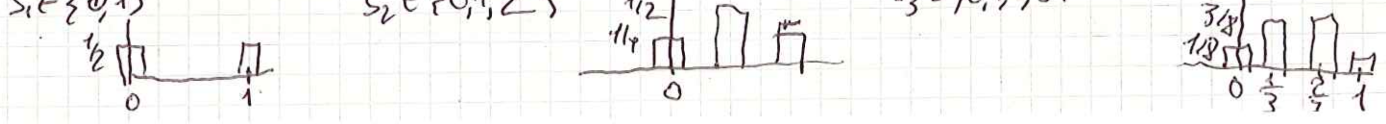
Ex. $X_n \sim b(1/2)$: $\mathbb{P}(X_n = 0) = \frac{1}{2} = P(X_n = 1)$ $\quad X(\text{Pile}) = 0$, $X(\text{face}) = 1$

$\to \mathbb{E}(X) = \frac{0+1}{2}$, $Var(X) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$S_n = X_1 + \ldots + X_N \sim B(N, \frac{1}{2})$

$\frac{\sqrt{N}}{1/2}\left(\frac{S_N}{N} - \frac{1}{2} \right) \Longrightarrow N(0,1)$, $\quad \frac{S_N}{N} \sim N\left(\frac{1}{2}, \frac{1}{4N}\right)$

$S_1 \in \{0, 1\}$ $\quad S_2 \in \{0, 1, 2\}$ $\quad S_3 = \{0, 1, 2, 3\}$



Def : Stochastic process : $X = (\Omega, F, (F_t)_{t\in T}, (X_t)_{t\in T}, \mathbb{P})$

Ω : universe, domain
F : σ-alg
$\mathbb{P}$: proba

$\left.\right\}$ $(\Omega, F, \mathbb{P})$ a probabilistic space.

. $F_t \subset F$ is the filtration ; $F_t = \bigcup_{s\leq t} X_t$: information at time t.
$= F_{t_1} \subset F_{t_2}$ $t_1 < t_2$

T is the index set : $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $[0,1]$, $\mathbb{T}$: torus, a manifold, etc.

Example : gaussian stochastic process : $\forall t_1, \dots t_k \in T$, $T = [0, L]$

$\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_k}\end{pmatrix}$ has gaussian law with $m(t) = \mathbb{E}(X_t)$ and
$\Gamma(s,t) = \cancel{\mathbb{E}[(X_t - m(t))(X_s - m(s))]}$
$= \text{Cov}(X_t, X_s)$

Ex $\mathbb{T} = \mathbb{N}$: discrete

$$\begin{cases} X_{n+1} = X_n + \beta_{n+1} \\ X_0 = 0 \end{cases} \quad \beta_n \sim N(0,1) \text{ and } \beta_n \text{ iid}$$

$$\to X_n = \sum_{i=1}^{n} \beta_j$$

$\Rightarrow X$ is Gaussian since $\forall n$ $\begin{pmatrix} X_1 \\ \vdots \\ X_n\end{pmatrix}$ has gaussian law :

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{i} \beta_j\right) = \sum_{i=1}^{n} \underbrace{b_j \beta_j}_{\sim N(0, b_j^2)} \sim N(0, \sum_{i=1}^{n} b_i^2)$$

$\mathbb{E}(X_n) = 0$ since $\beta_i \sim N(0,1)$

$$\text{Cov}(X_n, X_m) = \sum_{j_1=1}^{n} \sum_{j_2=1}^{m} \underbrace{\text{Cov}(\beta_{j_1}, \beta_{j_2})}_{\delta_{j_1 j_2}} = \min(n, m)$$

$$\boxed{\text{Cov}(X_n, X_m) = \min(n,m)}$$

$X \sim N(m, \sigma)$
$aX \sim N(am, a^2\sigma^2)$

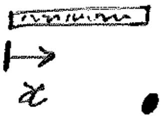
2. BROWNIAN MOTION (B.M.)

1. HISTORICAL INTRODUCTION

1827 Scottish botanist ROBERT BROWN observed the erratic behavior of tiny particles ejected from pollen grains suspended in water. Today, we explain this phenomenon by random collisions of surrounding molecules against the observed pollen grains.

1905 ALBERT EINSTEIN published the article at the foundation of modern brownian motion. We are going to study its physical origins to understand the meaning of the mathematical definition that we will give later.

2. INTERPRETATION of B.M. as DIFFUSION PROCESS



- Consider a long thin tube filled with some fluid at rest, and tiny particles that move inside it (e.g., ink particles).
- Let $f(x,t)$ be the particle density at time t in the point x. We consider 1D spatial motion to simplify.
- We set $\rho_x(y,\tau)$ the PROBABILITY DENSITY FUNCTION that a particle placed at x moves to $x+y$ in a time τ.

INCREMENT IN PARTICLE POSITION IS REGARDED AS A RANDOM VARIABLE

We get then:

$$f(x,t+\tau) = \int_{-\infty}^{+\infty} \underbrace{f(x-y,t)}_{\text{CHANGE IN PARTICLE DENSITY}} \rho_x(y,\tau)\,dy$$

- Taking a Taylor series of $f(x-y,t)$:

$$f(x,t+\tau) = f(x,t)\underbrace{\int_{-\infty}^{+\infty}\rho_x(y,\tau)\,dy}_{=1,\text{ because it is a pdf}} + \frac{\partial f}{\partial x}(x,t)\underbrace{\int_{-\infty}^{+\infty} -y\,\rho_x(y,\tau)\,dy}_{\text{by symmetry, } \rho_x(y,\tau)=\rho_x(-y,\tau)} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,t)\int_{-\infty}^{+\infty} y^2\rho_x(y,\tau)\,dy + \ldots$$

- So we are left with:

$$f(x,t+\tau) = f(x,t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,t)\int_{-\infty}^{+\infty} y^2 \rho_x(y,\tau)\,dy + \ldots$$

$$\lim_{\tau\to 0}\left(\frac{f(x,t+\tau)-f(x,t)}{\tau}\right)=\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,t)\times\lim_{\tau\to 0}\int_{-\infty}^{+\infty} y^2\,\frac{S_x(y,\tau)}{\tau}\,dy$$

Assumptions: (i) The other terms of the Taylor expansion are negligible;
(ii) The limit on the RHS exists finite & we denote its value by D.

- We get:

$$\frac{\partial f}{\partial t}(x,t)=\frac{D}{2}\frac{\partial^2 f}{\partial x^2}(x,t)$$, which is a DIFFUSION EQUATION !

Its solution is: $$f(x,t)=\frac{1}{\sqrt{2\pi D t}}\,e^{-\frac{x^2}{2Dt}}.$$

- WE DEDUCE THAT THE PARTICLE DENSITY $f(x,t)$ IS THE PROBABILITY DENSITY FUNCTION of A RANDOM VARIABLE WHICH FOLLOWS A NORMAL DISTRIBUTION $\mathcal{N}(0,Dt)$!

(According to these observations, we define our B.M. as follows:)

3. DEFINITION of B.M.

A stochastic process W is called BROWNIAN MOTION (or Wiener process) if:

- $W(0)=0$;
- $\forall\ 0\leq s\leq t,\quad W(t)-W(s)\sim\mathcal{N}(0,t-s)$; (THE INCREMENTS FOLLOW A NORMAL DISTRIBUTION)
- $\forall\ 0<t_1<t_2<\ldots<t_n$, The increments $W(t_1), W(t_2)-W(t_1),\ldots, W(t_n)-W(t_{n-1})$ are independent.

SIMULATION? plot_BM.m

RK (i) Another physical observation: A B.M. DOES NOT DEPEND ON ITS PAST, IT EVOLVES RANDOMLY. If we zoom on a B.M. and change the origin of our reference system, we find another B.M..

(ii) B.M. IS AN EXISTING OBJECT. To build one, we can take an ORTHONORMAL BASIS of L^2 (e.g. built upon Haar wavelets), then study series whose terms are these "primitives" multiplied by random variables. One can check that we get a B.M. under certain hypotheses...
The first explicit construction of this type is due to N. Wiener.

4. CONSTRUCTION of B.M.

- LET $(Y_j)_{j\in\mathbb{N}}$ BE A SEQUENCE of I.I.D. RANDOM VARIABLES, $\mathcal{N}(0,1)$.
- N. WIENER SHOWED THAT THE PROCESS $W = (W_t)_{t\in[0,\pi]}$ DEFINED BY:

$$W_t = \frac{t}{\sqrt{\pi}} Y_0 + \sum_{n\geq 1}\left(\sum_{j=2^{n-1}}^{2^n-1} \sqrt{\frac{2}{\pi}}\,\frac{\sin(jt)}{j}\, Y_j\right)$$

IS A BROWNIAN MOTION ON $I = [0,\pi]$.

- BASIS of $L^2([0,\pi])$: $e_0(t) = \frac{1}{\sqrt{\pi}}$, $e_n(t) = \sqrt{\frac{2}{\pi}}\cos(nt)$
(EXPANSION of WHITE NOISE IN THIS BASIS) $(n\geq 1)$

By formally deriving:

$$\text{“}\quad \dot{W}_t = \frac{dW}{dt}(t) = \frac{1}{\sqrt{\pi}} Y_0 + \underbrace{\sum_{n\geq 1}\left(\sum_{j=2^{n-1}}^{2^n-1}\sqrt{\frac{2}{\pi}}\cos(jt)\, Y_j\right)}_{\text{divergent series almost everywhere}}\text{”}$$

$\longrightarrow$ WHITE NOISE.

⌈This example of Wiener was also the starting point of the theory of RANDOM FOURIER SERIES.⌋

RK: Recall from Guillaume's talk that the formal time derivative $\dot{W}(t) = \xi(t)$ is "1D white noise". (SAMPLE POINT) $\mapsto \omega\in\Omega$, ← UNIVERSE

However, we will see later, for almost every ω, the sample path $t\longmapsto W(t,\omega)$ is in fact differentiable for no time $t\geq 0$. Thus $\dot{W}(t) = \xi(t)$ does not really exist.

5. B.M. IS A CENTERED GAUSSIAN PROCESS

Guillaume has already defined what is a Gaussian process.

Def. $(X(t))_{t\in I}$, real process, is called Gaussian if $\forall k \geq 1$, and $t_1, \dots, t_k \in I$, $(X(t_1), \dots, X(t_k))$ is a Gaussian R.V..

Def. $(X(t))_{t\in I}$, real process, is centered if, $\forall t \in I$, $\mathbb{E}(X(t)) = 0$.

LEMMA. Suppose $W(t)$ is a (1D) B.M. Then:

(i) W is a centered Gaussian process (eq.(6), p.43)

$$\mathbb{E}(W(t)) = 0, \quad \mathbb{E}(W^2(t)) = t, \quad (t \geq 0)$$

(ii) Moreover, $\mathbb{E}(W(t)W(s)) = \min\{s, t\}$, $(t, s \geq 0)$.

cfr. $\operatorname{cov}(X\,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0$ (see Guillaume's)

Proof. (i) is immediate from the def. of B.M., since $W(t)$ is $\mathcal{N}(0,t)$.

(ii) Assume $t \geq s \geq 0$. Then

$$\mathbb{E}(W(s)W(t)) = \mathbb{E}\big((W(s) + W(t) - W(s))\,W(s)\big)$$

$$= \underbrace{\mathbb{E}(W^2(s))}_{W(s) \text{ is } \mathcal{N}(0,s)} + \underbrace{\mathbb{E}\big((W(t) - W(s))\,W(s)\big)}_{W(t)-W(s) \text{ is indep. of } W(s) \text{ (because of def. of. B.M.)}}$$

$$= s + \underbrace{\mathbb{E}(W(t) - W(s))}_{=0}\,\underbrace{\mathbb{E}(W(s))}_{=0} \quad \text{because they are centered}$$

$$= s = \min\{s, t\}.$$

$\square$

④ **6. SAMPLE PATH PROPERTIES:** (p. 53 Evans)

orally
We'll show that, for almost every ω, the SAMPLE PATH

$$t \mapsto W(t,\omega)$$

is uniformly Hölder continuous for each exponent $0<\gamma<\frac{1}{2}$, but nowhere Hölder continuous with any exponent $\gamma>\frac{1}{2}$.

Def. (uniformly Hölder continuous)
Let $0<\gamma\leq 1$. A function $f:[0,T]\to\mathbb{R}$ is called uniformly Hölder continuous with exponent $\gamma>0$ if there exists a constant K s.t.

$$|f(t)-f(s)|\leq K\,|t-s|^{\gamma} \qquad \forall s,t\in[0,T]$$

NB: $\gamma=1$ lipschitz continuity ➡ ~~almost~~ differentiable almost everywhere (Chain of inclusions)

SIMULATION? NO.

Thm. (continuity of Brownian sample paths) [P 16R in basso.]
For almost all ω and any $T>0$, the sample path $t\mapsto W(t,\omega)$ is uniformly Hölder continuous on $[0,T]$ for each exponent $0<\gamma<1/2$.

* ~~Proof (?) "ideas of proof are better"~~

[p. 54 Evans, corso Gilles p. 18R–19]

ORALLY
NOWHERE DIFFERENTIABILITY. Sample paths of Brownian motion are with probability one nowhere Hölder continuous with exponent $\gamma>\frac{1}{2}$, and thus are nowhere differentiable.

Thm. (i) For each $\frac{1}{2}<\gamma\leq 1$ and almost every ω, $t\mapsto W(t,\omega)$ is NOWHERE HÖLDER CONTINUOUS with exponent γ. $\left(\mathbb{P}(\{W\in C^{\gamma}([0,T])\})=0 \text{ for } \gamma>\frac{1}{2}\right)$

(ii) In particular, for a.e. ω, the sample path $t\mapsto W(t,\omega)$ is NOWHERE DIFFERENTIABLE [~~and is of~~] ~~INFINITE VARIATION on each subinterval~~ $\left(\mathbb{P}(\{W\in BV(0,T)\})=0\right)$ ∎

Proof (~~TIME PERMITS ?~~) ~~MARKOV PROPERTY ??~~

3. $\boxed{\text{STOCHASTIC INTEGRALS}}$ (dovrebbe durare un'ora)

Remember what we want to do: WE WANT TO DEVELOP A THEORY of STOCHASTIC DIFFERENTIAL EQUATIONS of THE FORM

$$\begin{cases} dX = b(X,t)\,dt + B(X,t)\,dW \\ X(0) = X_0, \end{cases}$$

which we will interpret to mean:

$$(*) \qquad X(t) = X_0 + \int_0^t b(X,s)\,ds + \underline{\underline{\int_0^t B(X,s)\,dW}}$$

$\forall t \geq 0$.

We must first DEFINE STOCHASTIC INTEGRALS of THE FORM:

$$\boxed{\int_0^T G(s)\,dW(s)}$$

for some wide class of stochastic processes G.
(then the RHS of $(*)$ at least make sense).
But ..... the expression "$\int_0^T G(s)\,dW(s)$" simply cannot be understood as an ordinary integral.

[p.61 4.1.2.] **1. Riemann sums.**

What might be an appropriate definition for $\boxed{\int_0^T W\,dW =}$ (where $W(\cdot)$ is a one dimensional B.M.)

A reasonable procedure is to construct a Riemann sum approximation and then — if possible — to pass to limits.

~~We start by CONSIDERING~~ (→ indicator function)

$$\cancel{g(\omega,t) = \sum_{k=0}^{m-1} g_k(\omega)\, \mathbb{1}_{]t_k,t_{k+1}]}(t), \quad t\in[0,T]}$$

(P.66, 4.2.2)

~~$0 < t_1 < \dots < t_m = T$ partition of $[0,T]$~~ $\omega\in\Omega$

~~WE SET~~ $\cancel{\int_0^T g(s)\,dW(s) := \sum_{k=0}^{m-1} g_k\big(W(t_{k+1}) - W(t_k)\big)}$

~~This is the Itô stochastic integral of G on the interval $(0,T)$.~~ $(*)$

~~g_k Random variables~~

DEF. (i) $[0,T]$ interval, partition P of $[0,T]$ is
$$P := \{0 = t_0 < t_1 < \dots < t_m = T\}.$$

(ii) mesh size of P is
$$|P| := \max_{0 \le k \le m-1} |t_{k+1} - t_k|.$$

(iii) For fixed $0 \le \lambda \le 1$ and P, set
$$\tau_k := (1-\lambda) t_k + \lambda t_{k+1} \qquad (k = 0, \dots, m-1)$$
(PARAMETRIZED SEGMENT: τ_k IS A POINT LYING WITHIN THE SUBINTERVAL $[t_k, t_{k+1}]$.)

DEF. For P & $0 \le \lambda \le 1$, we write

$W(\tau_k)$

$$R = R(P, \lambda) := \sum_{k=0}^{m-1} W(\tau_k)\left(W(t_{k+1}) - W(t_k)\right)$$

$t_k \quad \tau_k \quad t_{k+1}$

(This is the Riemann sum approximation of $\int_0^T W \, dW$.)

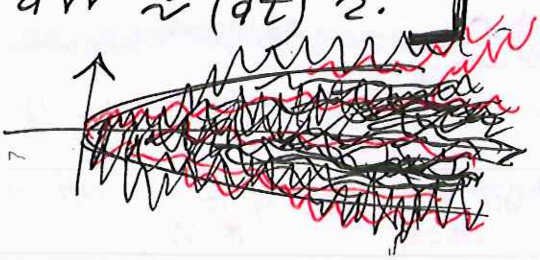
I will not talk about this in this minicourse, but different choices of λ lead to different definitions of the stochastic integral. $\lambda = 0$ ITÔ, $\lambda = 1/2$ STRATONOVICH.

Q: What happens when $|P| \to 0$, with λ fixed?

LEMMA (quadratic variation). (P.62) Let $[a,b]$ be an interval in $[0,\infty)$, and suppose
$$P_n := \{a = t_0^{(n)} < t_1^{(n)} < \dots < t_{m_n}^{(n)} = b\}$$
are partitions of $[a,b]$, with $|P_n| \to 0$ as $n \to \infty$. Then
$$\sum_{k=0}^{m_n - 1} \left(W(t_{k+1}^{(n)}) - W(t_k^{(n)})\right)^2 \longrightarrow (b-a) \text{ in } L^2(\Omega),$$
as $n \to \infty$.

(length of the interval)

RK. This result partly justifies the heuristic idea that
$$dW \approx (dt)^{1/2}.$$



SIMULATION

plot_BM.m

MAYBE STOP HERE ???

(4.2) [p. 65 EVANS] **2. Itô's Integral** (5) ter

Let $W(\cdot)$ be a 1D B.M. defined on some probability space $(\Omega, \mathcal{U}, \mathbb{P})$.

INFORMAL DISCUSSION of NONANTICIPATING PROCESSES

Informal def 1 (idea)

$\mathcal{F}(\cdot)$, family of σ-algebras, called nonanticipating (w.r.t. $W(\cdot)$) (or filtration) if $\mathcal{F}(t)$ "contains all information available to us at time t", but not the information after t!

Informal def 2 A real-valued stoch. process $G(\cdot)$ is called NONANTICIPATING (w.r.t. $\mathcal{F}(\cdot)$) if for each time $t \geq 0$, $G(t)$ is $\mathcal{F}(t)$-measurable.

(idea: "$\forall t \geq 0$, the R.V. $G(t)$ depends upon only the information available in the σ-algebra $\mathcal{F}(t)$.")

STRONGER NOTION [progressively measurable] stochastic processes. → THE IDEA IS THAT $G(\cdot)$ IS NONANTICIPATING AND, IN ADDITION, IS JOINTLY MEASURABLE IN THE VARIABLES t AND ω TOGETHER.

~~MEASURE THEORETIC ISSUES CAN BE CONFUSING~~

ORALLY

Idea: for ~~nonanticipating~~ progressively measurable processes, we can define, understand, the Itô stochastic integral $\int_0^T G\,dW$ in terms of useful and elegant formulas.

→ È quello che Gilles ha denotato L_p?

Def. (i) $\mathbb{L}^2(0,T)$ denote the space of all real-valued, ~~nonanticipating~~ "progressively measurable" stochastic processes $G(\cdot)$ s.t. $\mathbb{E}\left(\int_0^T G^2\,dt\right) < \infty$.

(ii) $\mathbb{L}^1(0,T)$ .... $F(\cdot)$, s.t. $\mathbb{E}\left(\int_0^T |F|\,dt\right) < \infty$.

(4.2.2.) **STEP PROCESSES.** A process $G \in \mathbb{L}^2(0,T)$ is called a STEP PROCESS if $\exists$ partition $P = \{0 = t_0 < t_1 < \dots < t_m = T\}$ of $[0,T]$ s.t. $G(t) \equiv G_k$ for $t_k \leq t < t_{k+1}$ $(k = 0, \dots, m-1)$: DRAWING

R.V.

$$G(\omega, t) = \sum_{k=0}^{m-1} G_k(\omega)\, \mathbb{1}_{]t_k, t_{k+1}]}(t), \quad t \in [0,T], \ \omega \in \Omega.$$

NOW

DEF. Let $G \in \mathcal{E}$. ~~be a step process in ...~~ We SET

$$(*) \quad \int_0^T G\,dW := \sum_{k=0}^{m-1} G_k \left(W(t_{k+1}) - W(t_k)\right).$$

THIS IS THE ITÔ'S STOCHASTIC INTEGRAL of G ON THE INTERVAL $(0,T)$.

(NOTE CAREFULLY THAT THIS INTEGRAL IS A R.V.)

5 quater

We denote by $\mathcal{E}$ the set of step processes:

$$\mathcal{E} = \{ \text{step processes} \}.$$

$\mathbb{L}^2(0,T)$ For $X, Y \in \mathbb{L}^2(0,T)$, let us define the distance function: $d_{\mathbb{L}^2}(X,Y) := \mathbb{E}\left[\int_0^T |X-Y|^2 dt\right]$.

$(\mathbb{L}^2, d_{\mathbb{L}^2})$ is a complete metric space.

(APPROXIMATION BY STEP PROCESSES)

"LEMMA" The set $\mathcal{E}$ of step processes is dense in $(\mathbb{L}^2, d_{\mathbb{L}^2})$, i.e., there exists a sequence of step processes that converges to an element G of $\mathbb{L}^2(0,T)$: $(G_n)_{n \in \mathbb{N}} \in \mathcal{E}$

(this is the same as lemma p. 68 Evans)

$$d_{\mathbb{L}^2}(G, G_n) \xrightarrow[n \to +\infty]{} 0.$$

(LEMMA p. 66 IN BASSO).

LEMMA Assume that g_k is independent of $\{W(t_{\ell+1}) - W(t_\ell);\ \ell \geq k\}$

Mean: $\boxed{\mathbb{E}\left(\int_0^T g(s)\,dW(s)\right)} \overset{\text{using } *}{=} \sum_{k=0}^{n-1} \mathbb{E}\left(g_k\left(W(t_{k+1}) - W(t_k)\right)\right)$

indep. hp.

$= \sum_{k=0}^{n-1} \mathbb{E}(g_k)\, \underbrace{\mathbb{E}\left(W(t_{k+1}) - W(t_k)\right)}_{\text{By def. of B.M.} = 0} = 0$

Variance: $\boxed{\mathbb{E}\left(\left(\int_0^T g(s)\,dW(s)\right)^2\right)} = \sum_{k,\ell=0}^{n-1} \mathbb{E}\left(g_k g_\ell \left(W(t_{k+1}) - W(t_k)\right)\left(W(t_{\ell+1}) - W(t_\ell)\right)\right)$

$= \sum_{k=0}^{n-1} \mathbb{E}\left(\underline{g_k^2}\,\underline{\left(W(t_{k+1}) - W(t_k)\right)^2}\right) +$

$\mathbb{E}((W_t - W_s)^2) = t - s$ (Gaussian process)

THIS TERM IS INDEPENDENT of $g_k g_\ell (W(t_{k+1}) - W(t_k))$

$+\, 2 \sum_{k<\ell} \mathbb{E}\left(g_k g_\ell \left(W(t_{k+1}) - W(t_k)\right)\left(W(t_{\ell+1}) - W(t_\ell)\right)\right)$

indep. $= \sum_{k=0}^{n-1} \mathbb{E}(g_k^2)(t_{k+1} - t_k) + 2\sum_{k<\ell} \mathbb{E}\left(g_k g_\ell \left(W(t_{k+1}) - W(t_k)\right)\right) \underbrace{\mathbb{E}\left(W(t_{\ell+1}) - W(t_\ell)\right)}_{0}$

(By def of B.M.)

$$\boxed{=: \int_0^T \mathbb{E}\left(g(s)^2\right) ds.}$$

* We actually "built" "have" an isometry for the stochastic integral.

THE PLAN NOW IS TO APPROXIMATE AN ARBITRARY PROCESS $G \in \mathbb{L}^2(0,T)$ BY STEP PROCESSES IN $\mathbb{L}^2(0,T)$ AND, THEN PASS TO LIMITS TO DEFINE THE ITÔ INTEGRAL of G.

space of all "real-valued, progressively measurable stochastic processes s.t. $\mathbb{E}\left(\int_0^T G^2\,dt\right) < \infty$"

p. 68 LEMMA (Approximation by step processes). If $G \in \mathbb{L}^2(0,T)$, there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of bounded step processes $G_n \in \mathbb{L}^2(0,T)$ such that

$$\mathbb{E}\left(\int_0^T |G - G_n|^2\,dt\right) \xrightarrow[n \to +\infty]{} 0.$$

(This is exactly what I defined as distance

$d_{\mathbb{L}^2}(G, G_n) \to 0$.)

(dense)

distance de cet espace

[p.68 § 4.2.3]

⑦

DEFINITION OF ITÔ'S INTEGRAL & PROPERTIES.

If $G \in \mathbb{L}^2(0,T)$, take step processes G_n as above.

Then (QUI NON SIAMO IN UNO SPAZIO COMPLETO) SPAZIO COMPLETO $(\mathbb{L}^2, d_{\mathbb{L}^2})$

$$\mathbb{E}\left(\left(\int_0^T G_n - G_m \, dW\right)^2\right) = \underbrace{\mathbb{E}\left(\int_0^T (G_n - G_m)^2 dt\right)}_{= d_{\mathbb{L}^2}(G_n, G_m)} \to 0$$

as $n, m \to \infty$, and so the limit

ITÔ'S INTEGRAL

$$\int_0^T G \, dW := \lim_{n \to \infty} \int_0^T G_n \, dW$$

exists in $L^2(\Omega)$.

> Remark: the sequence $\int_0^T G_n \, dW$ converges and the limit does not depend upon the particular sequence of step process approximations $(G_n)_{n \in \mathbb{N}}$ in $\mathbb{L}^2(0,T)$.

THEOREM: For all $G, H \in \mathbb{L}^2(0,T)$, we have:

① $\mathbb{E}\left(\int_0^T G(s) \, dW(s)\right) = 0$.

② $\mathbb{E}\left(\left(\int_0^T G(s) \, dW(s)\right)^2\right) = \mathbb{E}\left(\int_0^T G^2(s) \, ds\right)$ (**ITÔ'S** ISOMETRY)

Gallardo, corol. 4.3.2 pg. 123 → ③ $\mathbb{E}\left(\int_0^T G(s) \, dW(s) \int_0^T H(s) \, dW(s)\right) = \mathbb{E}\left(\int_0^T G(s) H(s) \, ds\right)$.

("COVARIANCE of 2 STOCHASTIC INTEGRALS")

Proof idea: • ① & ② are consequences of the similar rules for step processes.

• ③ results from ② and the identity $2ab = (a+b)^2 - a^2 - b^2$.

NO

$$\frac{1}{2}\mathbb{E}(2ab) = \frac{1}{2}\mathbb{E}\left((a+b)^2 - a^2 - b^2\right) =$$

$$= \frac{1}{2}\mathbb{E}\left(\left(\int_0^T (G+H) dW\right)^2\right) - \frac{1}{2}\mathbb{E}\left(\left(\int_0^T G \, dW\right)^2\right) - \frac{1}{2}\mathbb{E}\left(\left(\int_0^T H \, dW\right)^2\right) =$$

using ②

$$= \frac{1}{2}\mathbb{E}\left(\int_0^T \underbrace{(G+H)^2}_{G^2+H^2+2GH} ds\right) - \frac{1}{2}\mathbb{E}\left(\int_0^T G^2 ds\right) - \frac{1}{2}\mathbb{E}\left(\int_0^T H^2 ds\right) =$$

$$= \mathbb{E}\left(\int_0^T G H \, ds\right) \blacksquare$$

(4.3) 3. ITÔ'S CHAIN & PRODUCT RULE

Def. Suppose that $X(\cdot)$ is a real-valued stochastic process satisfying

$$X(r) = X(s) + \int_s^r F\,dt + \int_s^r G\,dW$$

for some $F \in \mathbb{L}^1(0,T)$, $G \in \mathbb{L}^2(0,T)$ and all times $0 \le s \le r \le T$. We say that $X(\cdot)$ has the stochastic differential

$$dX = F\,dt + G\,dW, \qquad \text{for } 0 \le t \le T.$$

ORALLY

(NB) the symbols are simply an abbreviation for the integral expressions above; strictly speaking, "dX", "dt" and "dW" have NO MEANING ALONE.

Thm. Suppose that $X(\cdot)$ (real-valued stochastic process) has a stochastic differential

$$dX = F\,dt + G\,dW,$$

(F, G processes $F \in \mathbb{L}^1(0,T)$, $G \in \mathbb{L}^2(0,T)$).

Let $\phi : \mathbb{R} \times [0,T] \to \mathbb{R}$, $\phi(x,t)$, $\phi \in C^2$ (i.e., $\partial\phi/\partial t$, $\partial\phi/\partial x$, $\partial^2\phi/\partial x^2$ exist & are continuous).

Then $\phi(X(t),t)$ has the STOCHASTIC DIFFERENTIAL:

$$d\phi(X,t) = \phi_t\,dt + \phi_x\,dX + \frac{1}{2}\phi_{xx}G^2\,dt$$
$$= \left(\phi_t + \phi_x F + \frac{1}{2}\phi_{xx}G^2\right)dt + \phi_x G\,dW.$$

(this is called ITÔ'S CHAIN RULE or ITÔ'S FORMULA).

(Remark: ϕ_x, ϕ_t, etc., above is $(X(t),t)$.).

EXAMPLES/ILLUSTRATIONS of ITÔ'S CHAIN RULE.

Ex. 1) [redacted] $X(\cdot) = W(\cdot)$, $\phi(x) = x^m$. LET US STUDY THE CASE ($dX = dW$). Thus $F \equiv 0$, $G \equiv 1$. Hence Itô's chain rule gives

$$d(W^m) = mW^{m-1}\,dW + \frac{1}{2}m(m-1)W^{m-2}\,dt.$$

10

So the particular case $m = 2$ reads:

$$d(W^2) = 2W\,dW + \underbrace{dt}_{\text{additional term of Itô}}$$

→ see dropbox SIMULATION OF THE INTEGRAL

COMPARE WITH $\cancel{d(W^2) = 2W\,dW}$ ~~(see also p. 24 Gilles)~~

This integrated is the identity $\boxed{2\int_0^1 W\,dW = W^2(1) - 1}$.

EX 2 ~~Another example of stochastic differential is:~~

Let's study another case: $dX = F\,dt + dW$

(This is example from pg. 24 of Gilles' course; it coincides with the result of theorem p. 73-74 of Evans, taking $X_1 = X_2 = X$, $F_1 = F_2 = F$, $G_1 = G_2 = 1$.)

Then

$$d(X^2) = 2X\,dX + \underbrace{dt}_{\text{ADDITIONAL TERM.}}$$

(~~PROOF IS QUITE LONG 1-2 pages~~ NO PROOF.)

$$\cancel{\frac{d(X^2)}{dX} = 2X \qquad d(X^2) = 2X\,dX}$$

This is an instance of Itô's product rule:

Thm (Itô's product rule) Suppose:

$$\begin{cases} dX_1 = F_1\,dt + G_1\,dW \\ dX_2 = F_2\,dt + G_2\,dW \end{cases} \qquad (0 \le t \le T)$$

for $F_i \in \mathbb{L}^1(0,T)$, $G_i \in \mathbb{L}^2(0,T)$ $(i = 1,2)$. Then

$$d(X_1 X_2) = X_2\,dX_1 + X_1\,dX_2 + \underbrace{G_1 G_2\,dt}_{\text{Itô's correction term}}$$

(Itô's product rule or Itô's formula).

This can be generalised to have more X_i ......

[END of MY PART]

Partie III : EDS et analyse numérique stochastique

III - EDS

idée : $X'(t) = f(t, X(t)) + g(t, X(t)) \dfrac{dW(t)}{dt}$

↳ formulation intégrale : $\boxed{X(t) = X(0) + \int_0^t f(s, X(s))\, ds + \int_0^t g(s, X(s))\, dW(s)}$ ⊛

Pb : ⊛ a-t-il une solution ?

c'est $f(s, \omega, X(s,\omega))$ mais... we will forget it.

1) Existence et Unicité

Thm : $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$

$g : [0,T] \times \mathbb{R}^n \to \mathcal{M}_{n,m}(\mathbb{R})$

W brownian motion in $\mathbb{R}^m$.

1) $|f(t, x_1) - f(t, x_2)| \leq L\, |x_1 - x_2|$

2) $|g(t, x_1) - g(t, x_2)| \leq L\, |x_1 - x_2|$

3) $|f(t, x)| \leq L(1 + |x|)$

4) $|g(t, x)| \leq L(1 + |x|)$

5) $\mathbb{E}[|X_0|^2] < +\infty$.

then $\exists!\, X \in \mathbb{L}^2([0,T])$ solution of ⊛.

Rq : unicity means if X_1, X_2 solutions $\mathcal{C}^0$ of ⊛ then

$$\mathbb{P}(X_1(t) = X_2(t),\ \forall t \in [0,T]) = 1$$

idea of proof : (Unicity) X_1, X_2 solutions of ⊛ then

$$X_1(t) - X_2(t) = \int_0^t (f(s, X_1) - f(s, X_2))\, ds + \int_0^t g(s, X_1) - g(s, X_2)\, dW(s)$$

$$\mathbb{E}[|X_1(t) - X_2(t)|^2] \leq 2 \times \mathrm{I} + 2 \times \mathrm{II}$$

$$\mathrm{I} \leq \mathbb{E}\left[\left|\int_0^t f(s, X_1) - f(s, X_2)\, ds\right|^2\right]$$

$$\underset{CS}{\leq} t\, \mathbb{E}\left[\int_0^t |f(s, X_1) - f(s, X_2)|^2\, ds\right] \leq T L^2 \int_0^t \mathbb{E}[|X_1 - X_2|^2(s)]\, ds$$

$$II = \mathbb{E}\left[\left|\int_0^t (g(s,X_1) - g(s,X_2))\,dW\right|^2\right] \quad (2)$$

$$= \mathbb{E}\left[\int_0^t |g(s,X_1) - g(s,X_2)|^2\,ds\right]$$

isométrie Itô

$$\leq L^2 \int_0^t \mathbb{E}[|X_1 - X_2|^2(s)]\,ds$$

donc $\boxed{\mathbb{E}[|X_1 - X_2|^2(t)] \leq C \int_0^t \mathbb{E}[|X_1 - X_2|^2(s)]\,ds}$

càd $\phi(t) \leq C \int_0^t \phi(s)\,ds$

Gronwall !!!

$\hookrightarrow$ si $\phi(t) \leq K + \int_0^t \overbrace{k(s)}^{\geq 0 \text{ et } \mathcal{C}^0} \phi(s)\,ds$

alors $\phi(t) \leq K \exp\left(\int_0^t k(s)\,ds\right)$

donc ici $\phi(t) \leq 0$

donc $\mathbb{E}[|X_1 - X_2|^2(t)] = 0 \ \forall t$

"donc $X_1(t) = X_2(t)$ p.s, $\forall t \in [0,T]$

On conclut en disant X_1, X_2 $\mathcal{C}^0$."

(b) Existence

On pose $\begin{cases} X^0(t) = X_0 \\ X^{n+1}(t) = X_0 + \int_0^t f(s, X^n(s))\,ds + \int_0^t g(s, X^n(s))\,dW(s) \end{cases}$

On pose $d^n(t) = \mathbb{E}[|X^{n+1}(t) - X^n(t)|^2]$

Par récurrence, on montre que $d^n(t) \leq \dfrac{(Mt)^{n+1}}{(n+1)!}$ (mêmes inégalités que dessus)

Doob donne alors

$$\mathbb{E}\left[\max_{0\le t\le T} |X^{n+1} - X^n|^2(t)\right] \le C\frac{(MT)^{n+1}}{(n+1)!}$$

Markov donne $\mathbb{P}(|X|>t) \le \dfrac{\mathbb{E}[|X|]}{t}$

donc ici $\mathbb{P}\left(\max_{[0,T]} |X^{n+1}(t) - X^n(t)| > \frac{1}{2^n}\right) \le 2^{2n}\, \dfrac{C(MT)^{n+1}}{(n+1)!}$

Or $\displaystyle\sum_{n=1}^{+\infty} 2^{2n}\frac{(MT)^{n+1}}{(n+1)!} < +\infty$ donc Borel Cantelli conclut :

$$\mathbb{P}\left(\limsup_n \left\{\max_{[0,T]} |X^{n+1}(t) - X^n(t)|^2 > \frac{1}{2^n}\right\}\right) = 0$$

c.à.d $X^n = X^0 + \sum_{j=0}^{n-1} X^{j+1} - X^j$ CV uniformément p.s. vers X.

On peut passer à la limite dans ~~ensuite on peut~~

$$X^n = X^0 + \int_0^t f(s, X^n(s))\,ds + \int_0^t g(s, X^n(s))\,dW(s)$$

et cela donne X solution de ⊛.

→ we have to check also $X \in \mathbb{L}^2$ . . . .

$\square$

2) Explicitely solvable SDEs and examples

→ in general not solvable explicitely

ⓐ Linear SDEs

La solution de $dX = a(t)X\,dt + b(t)X\,dW$ est

$$X(t) = \exp\Big(\underbrace{\int_0^t a(s)\,ds + \int_0^t b(s)\,dW(s) - \tfrac{1}{2}\int_0^t b(s)^2\,ds}_{Y(t)}\Big)X_0$$

En effet, $dY(t) = a(t)\,dt + b(t)\,dW(t) - \frac{1}{2}b^2(t)\,dt$

et $dX(t) = e^{Y(t)} X_0 \, dY(t) + \frac{1}{2} e^{Y(t)} X_0 \, b^2(t) \, dt$

$$= \underbrace{e^{Y(t)} X_0}_{= X(t)} \left(a(t) \, dt + b(t) \, dW(t) \right)$$

OK

Ex: solution $dY = Y \, dW$ is $e^{W(t) - t/2} \, Y(0)$.

Application: Black-Scholes model / Bachelier

S_t : price of ~~a action~~ share at time t

$$\begin{cases} dS_t = \underbrace{\mu S_t \, dt}_{\text{drift}} + \underline{\sigma S_t \, dW_t} \\ S_0 = s_0 \end{cases}$$

c'à'd "$\underbrace{\frac{dS_t}{S_t}}_{\text{relative change of action}} = \underbrace{\mu \, dt}_{\text{drift of the price}} + \underbrace{\sigma \, dW_t}_{\text{volatility of share}}$"

$$\mathbb{E}[S_t] = \mathbb{E}[S_0] + \mu \int_0^t \mathbb{E}[S_s] \, ds$$

so $\underline{\mathbb{E}[S_t] = \underbrace{e^{\mu t}}_{>0} \mathbb{E}[S_0]}$

numerical approach.

(b) Brownian bridge

$B(0) = B(1) = 0$ and B looks like a Brownian motion.

that is $B_t \overset{\mathcal{L}}{=} (W_t \mid W_1 = 0)$, $t \in [0, T]$

then $\begin{cases} dB(t) = -\frac{B}{1-t}\,dt + dW & \text{works} \\ B(0) = 0 \end{cases}$ (5)

We check $B(t) = \exp\left(-\int_0^t \frac{1}{1-s}\,ds\right)\left(B(0) + \int_0^t \exp\left(\int_0^s \frac{1}{1-r}\,dr\right) dW(s)\right)$

$$= (1-t)\int_0^t \frac{1}{1-s}\,dW_s$$

OPM $\underline{B(1) = 0 \text{ p.s}}$

$\underline{\text{Or}}$ $B(t) = W(t) - t\,W(1)$ $\underline{\text{works}}$.

(c) Stochastic oscillator

$\underline{\text{idea:}}$ $\ddot{X} = -\lambda^2 X + \frac{dW}{dt}$?

$$\begin{cases} dX = Y\,dt \\ dY = -\lambda^2 X\,dt + dW \end{cases}$$

Then if $Y_0 = 0$, $\boxed{X(t) = X_0 \cos(\lambda t) + \frac{1}{\lambda}\int_0^t \sin(\lambda(t-s))\,dW_s}$

→ numerical experiment

(d) Molecular dynamic

A big particle (q,p) is submitted in a fluid to
- a potential V
- a friction force $-\gamma p$
- $\sqrt{\frac{2\gamma}{\beta}}\,dW$ collisions (as seen before)

then $\begin{cases} dq = p\,dt \\ dp = (-\nabla V(q) - \gamma p)\,dt + \sqrt{\frac{2\gamma}{\beta}}\,dW \end{cases}$ (Langevin equation)

$\gamma \to +\infty$: friction tends

acceleration $dq \to 0$

then $\begin{cases} dq = p\,dt \\ 0 = (-\nabla V(q) - \gamma p)\,dt + \sqrt{\frac{2\gamma}{\beta}}\,dW \end{cases}$

and $dq = -\gamma^{-1}\nabla V(q)\,dt + \sqrt{\frac{2}{\gamma\beta}}\,dW$

$\gamma \to +\infty$: $dX = -\nabla V(X)\,dt + \sigma\,dW$
(overdamped Langevin)

Simulation

(e) Stochastic Schrödinger.

$$i\,du + \Delta u\,dt + |u|^{2\sigma} u\,dt = 0 \quad \ldots$$

$$i\,du + \Delta u \circ d\beta + |u|^{2\sigma} u\,dt = 0 \quad \ldots$$

$$i\,du + \Delta u\,dt + |u|^{2\sigma} u\,dt + g(u)\,dW = 0 \ldots\ldots$$

VII - Numerical Analysis of SDEs

1) Strong error

to simplify

given W, I want X solution of $\begin{cases} dX = f(X)\,dt + g(X)\,dW \\ X(0) = X^0 \end{cases}$

Euler-Maruyama method

T $\qquad T/N = h$

N steps

$t_n = nh, \quad n = 0, \dots, N$

Algorithm:
$$\begin{cases} X_0 = X^0 \\ X_{n+1} = X_n + h f(X_n) + g(X_n) \underbrace{\Delta W_n}_{} \end{cases}$$
$$:= W(t_{n+1}) - W(t_n) \sim \sqrt{h}\, \mathcal{N}(0, Id)$$

Déf: (Mean-Square error)
$(X_n)_n$ has strong order $\gamma > 0$ if $\forall h$ small enough,
$$\mathbb{E}\left[|X_n - X(t_n)|^2\right]^{1/2} \leq C h^\gamma \text{ and } \underline{C \perp\!\!\!\perp h, n}.$$

Prop: EM has order $1/2$

→ Comparison deterministic Euler

proof: super hard

Milstein scheme:

$$X(t) = X_0 + \underbrace{\int_0^t f(X(s))\,ds}_{\approx f(X_0) + \mathcal{O}(t)} + \underbrace{\int_0^t g(X(s))\,dW(s)}_{\approx g(X_0)W(t) + \mathcal{O}(\sqrt{t})}$$

$$X(t) = X_0 + g(X_0)\,W(t) + \mathcal{O}(\sqrt{t})$$

$$\int_0^t g(X(s))\,dW(s) \approx g(X_0)\,W(t) + g'(X_0)(g(X_0))\int_0^t W(s)\,dW(s) + o(t)$$

so $$X_{n+1} = X_n + h f(X_n) + g(X_n)\Delta W_n + g'(X_n)(g(X_n))\underbrace{\int_{t_n}^{t_{n+1}} W(s)\,dW(s)}_{= \frac{1}{2}(\Delta W_n^2 - h)}$$

Prop: Milstein has strong order 1.

→ numerical experiments

2) Weak error

→ We do not know W, we want to know the law of X solution of

$$\begin{cases} dX = f(X)\,dt + g(X)\,dW \\ X(0) = X^0 \end{cases}$$

Déf: a method $(X_n)_n$ has weak order γ if $\forall \phi$ test function (usually $\mathcal{C}^k_P$ ...) /

and $\forall h$ small enough such that $T = Nh$,
$$|\mathbb{E}[\phi(X_n)] - \mathbb{E}[\phi(X(t_n))]| \leq C h^\gamma$$

ex: $\phi(x) = x \to \mathbb{E}$
$\phi(x) = x^2 \to \text{Var}$
Langevin $\mathbb{E}[(..)^2] =$ temperature

↳ ex invariant measure

→ stupid weak vs strong

Rq: We recall $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$ ....

Thm: EM has weak order 1: $\boxed{X_{n+1} = X_n + h f(X_n) + \sqrt{h}\, g(X_n) \overbrace{\xi_n}^{\sim \mathcal{N}(0, \mathrm{Id})}}$ ($\perp\!\!\!\perp n$)

Proof: (a) Lemma: If $\mathbb{E}[X_0^{2m}] < +\infty, \forall m$, $\mathbb{E}[X_n^{2m}] \leq C_m < +\infty$

Proof: $X_{n+1} = X_n + h f(X_n) + \sqrt{h}\, g(X_n) \xi_n$

• $|\mathbb{E}[X_{n+1} - X_n | X_n = x]| = |\mathbb{E}[h f(x)]| = |h f(x)| \leq C(1 + |x|) h$ (1)

• $|X_{n+1} - X_n| \leq h |f(X_n)| + \sqrt{h}\, |g(X_n)| |\xi_n|$
$\leq C(1 + |X_n|) h + C(1 + |X_n|) \sqrt{h} |\xi_n|$
$\leq C(1 + |X_n|) \sqrt{h} (\underbrace{\sqrt{h}}_{\leq \sqrt{T}} + |\xi_n|)$ and $\underbrace{\mathbb{E}[|\xi_n|^{2m}]}_{\perp\!\!\!\perp n} < +\infty, \forall m$

that is $|X_{n+1} - X_n| \leq M_n (1 + |X_n|) \sqrt{h}$ and $\mathbb{E}[M_n^{2m}] \leq \underbrace{C_m}_{\forall n}$ (2)

- Define $X_{n+1} - X_n = \Delta X_n$

$$(X_{n+1})^{2m} = |X_n + \Delta X_n|^{2m} = \underset{\text{I}}{X_n^{2m}} + \underset{\text{II}}{\binom{2m}{1} X_n^{2m-1} \Delta X_n} + \underset{\text{III}}{\sum_{j=2}^{2m} \binom{2m}{j} X_n^{2m-j} \Delta X_n^j}$$

II $\left|\mathbb{E}[(X_n)^{2m-1} \Delta X_n]\right| = \left|\mathbb{E}[(X_n)^{2m-1} \mathbb{E}[\Delta X_n | X_n]]\right|$

$$\overset{(1)}{\leq} \mathbb{E}[|X_n|^{2m-1} K(1+|X_n|)h]$$

$$\leq Kh(1+\mathbb{E}[|X_n|^{2m}])$$

III $\mathbb{E}[|X_n|^{2m-j} |\Delta X_n|^j] \overset{(2)}{\leq} \mathbb{E}[|X_n|^{2m-j} M_m{}^j (1+|X_n|^j) h^{j/2}]$

$$\leq Kh(1+\mathbb{E}[|X_n|^{2m}]) \quad \text{because } j \geq 2$$

Finally, $\mathbb{E}[|X_{n+1}|^{2m}] \leq (1+Kh)\,\mathbb{E}[|X_n|^{2m}]$

$$\leq e^{Kh}\,\mathbb{E}[|X_n|^{2m}]$$

$$\therefore \mathbb{E}[|X_n|^{2m}] \leq \underbrace{e^{Knh}}_{\leq e^{KT} < +\infty} \mathbb{E}[|X_0|^{2m}]$$

$\square$

b) Local error

$$\mathbb{E}[\phi(X_1)] = \mathbb{E}[\phi(x_0 + hf(x_0) + \sqrt{h}\, g(x)\xi)]$$

$$= \phi(x) + h\,\phi'(f) + \frac{h}{2} \underbrace{\mathbb{E}[\phi''(g\xi, g\xi)]}_{= \mathbb{E}[\xi^2]\,\phi''(g,g)} + O(h^2)$$

$$= \phi + h\mathcal{L}\phi + \text{“}O(h^2)\text{”}$$

where $\mathcal{L}\phi = \phi'(f) + \frac{1}{2}\phi''(g,g)$

Then $u(x,t) = \mathbb{E}_x[\phi(X(t))]$ is solution of Kolmogorov equation

$$\boxed{\frac{\partial u}{\partial t} = \mathcal{L}u \ , \ u(x,0) = \phi(x)}$$

$\rightarrow$ use Itô's formula

then $\mathbb{E}[\phi(X(h))] = \phi(x) + h\mathcal{L}\phi(x) + \frac{1}{2}h^2\mathcal{L}^2\phi(x) + \dots$

then $$\boxed{\left|\mathbb{E}[\phi(X_1)] - \mathbb{E}[\phi(X(h))]\right| \leq Ch^2(1+|x|^K)}$$

© Global error

$X^x(t)$ solution coming from x at $t=0$

X_n^x 〰

$$e = \mathbb{E}[\phi(X^x(t_N)) - \phi(X_N^x)]$$

$$= \sum_{i=1}^{N} \mathbb{E}[\phi(X^{X_{N-i}^x}(hi))] - \mathbb{E}[\phi(X^{X_{N-i+1}^x}(h(i-1)))]$$

$$= \sum_{i=1}^{N} \mathbb{E}[\phi(X^{X^{X_{N-i}^x}(h)}(h(i-1)))] - \mathbb{E}[\phi(X^{X_1^{X_{N-i}^x}}(h(i-1)))]$$

$\tilde{\Phi}_i(x) = \phi \circ X^x(h(i-1))$ test function ...

local error gives $\left|\mathbb{E}[\tilde{\Phi}_i(X^{X_{N-i}^x}(h))] - \mathbb{E}[\tilde{\Phi}_i(X_1^{X_{N-i}^x})]\right|$

$$\leq \mathbb{E}[Ch^2(1+|X_{N-i}^x|^K]$$

so $$|e| \leq \sum_{i=1}^{N} \mathbb{E}[Ch^2(1+|X_{N-i}^x|^K)]$$

$$\leq Ch^2 \sum_{i=1}^{N}(1+\underbrace{\mathbb{E}[|X_{N-i}^x|^K]}_{\leq C})$$

$\leq C$ because of bounded moment

$$\leq Ch^2 N \leq Ch$$

□

→ CV curve for weak E.M..

Example: $dX = aX\,dW - \frac{a^2}{2} X\,dt \quad , X(0) = 1$

$$X(1) = e^{aW(1)}$$

$$X_{n+1} = X_n + a X_n \sqrt{h}\,\xi_n - \frac{a^2}{2} X_n h = \left(1 + a\sqrt{h}\,\xi_n - \frac{a^2}{2} h\right) X_n$$

$$X_n = \prod_{i=1}^{n} \left(1 + \sqrt{h}\,a\,\xi_i - h\frac{a^2}{2}\right)$$

$$= 1 + \underbrace{a\sqrt{h}\sum_{i=1}^{n} \xi_i}_{\overset{\mathcal{L}}{=}\ W(nh)} + O(h)$$

Extension: Stochastic Runge-Kutta for $dX = f(x)\,dt + \sigma\,dW$

$$\begin{cases} Y_i^n = X_n + h\sum_{j=1}^{\hat{s}} a_{ij} f(Y_j^n) + d_i \sigma \sqrt{h}\,\xi_n \\ X_{n+1} = X_n + h\sum_{i=1}^{\hat{s}} b_i f(Y_i^n) + \sigma\sqrt{h}\,\xi_n \end{cases}$$

- bounded moments → OK
- local error → RK conditions: $\sum b_i = 1 \Rightarrow A_0 = \mathcal{L}$

$$\left.\begin{array}{l} \sum b_i c_i = 1/2 \\ \sum b_i d_i^2 = 1/2 \\ \sum b_i d_i = 1/2 \end{array}\right\} \Rightarrow A_1 = \frac{1}{2}\mathcal{L}^2$$

$$\mathbb{E}[\phi(X_1)] = \phi + hA_0\phi + h^2 A_1\phi + h^3 A_2\phi + \dots$$

III - Other uses of Stochastic Analysis in Numerical Analysis

a) The WOS algorithm

goal: Solve $\begin{cases} h \in \mathcal{C}(\overline{D}, \mathbb{R}) \cap \mathcal{C}^2(D, \mathbb{R}) \\ \Delta h = 0 \text{ in } D \\ h = f \text{ on } \partial D \end{cases}$ (D P)

where D is a bounded open set in $\mathbb{R}^d$ (d large).

Prop: u is harmonic in D iff u satisfies the mean-value property, i.e. $\forall\ \overline{B_r(x)} \subset D$,

$$u(x) = \frac{1}{\underbrace{\sigma(\partial B_r(x))}_{\text{measure of } \partial B_r(x)}} \int_{\partial B_r(x)} u(z)\, d\sigma(z)$$

Rq: $\simeq$ Cauchy formula for holomorphic functions
harmonic functions in $\mathbb{R}^2 \lneq$ holomorphic functions in $\mathbb{C}$, ...

Proof: ($\Rightarrow$) if $\Delta u = 0$, by Itô formula, where $W_0 = x$

$$u(W_t) = u(W_0) + \int_0^t \nabla u(W_s)\, dW_s + \frac{1}{2}\int_0^t \underbrace{\Delta u(W_s)}_{=0}\, ds$$

$$\mathbb{E}_x[u(W_t)] = \mathbb{E}_x[u(W_0)] = u(x)$$

If $\tau_{B_r(x)} = \inf\{t > 0,\ W_t \in \partial B_r(x)\}$,

with dominated CV thm,

$$\underbrace{E_x\left[u\left(W_{\tau_{B_r(x)}}\right)\right]} = u(x)$$

$$= \int_{\partial B_r(x)} u(z)\, d\mathbb{P}_{W_{\tau_{B_r(x)}}}(z) = \int_{\partial B_r(x)} u(z)\, \frac{d\sigma(z)}{\sigma(\partial B_r(x))}$$

because $W_{\tau_{B_r(x)}}$ follows a uniform law on $\partial B_r(x)$ by symmetry.

$\Leftarrow$ Analytic proof

differentiation under $\int$ gives u C^∞.

then for $y \in \partial B_r(0)$,

$$u(x+y) = u(x) + y \cdot \nabla u(x) + \frac{1}{2} D^2 u(x)(y,y) + o(r^2)$$

$$\int_{\partial B_r(0)} y_i \, d\sigma = 0 \quad \text{and} \quad \int_{\partial B_r(0)} y_i y_j \, d\sigma = 0 \ (i \neq j)$$

so
$$u(x) \underset{\substack{\uparrow \\ \text{mean value property}}}{=} \frac{1}{\sigma(\partial B_r(x))} \int_{\partial B_r(0)} u(x+y) \, d\sigma(y)$$

$$= u(x) \times 1 + 0 + \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(x) \frac{1}{\sigma(\partial B_r(0))} \int_{\partial B_r(0)} y_i^2 \, d\sigma(y) + o(r^2)$$

but then
$$\frac{1}{\sigma(\partial B_r(0))} \int_{\partial B_r(0)} y_i^2 \, d\sigma(y) = \frac{1}{d\,\sigma(\partial B_r(0))} \int_{\partial B_r(0)} \overbrace{\|y\|^2}^{= r^2} \, d\sigma(y)$$

$$= \frac{r^2}{d}$$

so
$$u(x) = u(x) + \frac{r^2}{2d} \Delta u(x) + o(r^2)$$

$$\Rightarrow \boxed{\Delta u(x) = 0}.$$

$\square$

Summary: we found: if h solution of $\Delta h = 0$

then $\forall x \in D$, $h(x) = \mathbb{E}\left[h\left(\underbrace{W_{\tau_{B_r(x)}}}_{\sim \mathcal{U}(\partial B_r(x))}\right)\right]$

Wos algorithm:
$$\boxed{\begin{cases} X_r^x(1) = x \\ X_r^x(n+1) = X_r^x(n) + r\, d(X_r^x(n), \partial D)\, U_n \end{cases}}$$

where $r \in]0,1]$ is fixed and $U_n \sim \mathcal{U}(S^d)$

is sphere unit

→ exp. numérique

Prop: • $(X_r^x(n))_n$ CV a.s. to $X_r^x(\infty) \in \partial D$ ("à l'œil")

• $(X_r^x(n))_n$ is a martingale w.r.t. $\mathcal{F}_n = \sigma(U_1, \dots, U_{n-1})$

Thm: let h be the solution of (DP),

then $h(x) = \begin{cases} f(x) & \text{if } x \in \partial D \\ \mathbb{E}\left[f(X_r^x(\infty))\right] & \text{if } x \in D \end{cases}$

ex: $D = \square$

$f(x,y) = e^x \sin(y)$

idea: when $d(X_r^x(n), \partial D) < \varepsilon$, → stop

and $X_r^x(\infty) \approx \Pi_{\partial D}(X_r^x(n))$

hard for weird grids sets D (fractal...)

Cons/Pros

Rk: slow algorithm for simple problems (finite diff/elements are better)

but if dim is huge, finite elements fail (slower and slower

and here the CV stays the same ($1/2$ for Monte-Carlo)

ⓑ WOMS

Same with "moving sphere" in space and time

↳ solves the heat equation

→ numerical experiment

ⓒ MLMC

↳ goal : reduce noise

ⓓ Stochastic geometric integration / preserving properties

Ex : Langevin → preserving the invariant measure

OU Stochastic oscillator

$$\begin{cases} dX = Y\,dt \\ dY = -\lambda^2 X\,dt + dW \end{cases}$$

alors $~~d\mathbb{E}~~$ $d\left(\frac{\lambda^2}{2}X^2 + \frac{1}{2}Y^2\right) = \lambda^2 XY\,dt + Y(-\lambda^2 X\,dt + dW) + \frac{dt}{2}$

donc $\mathbb{E}\left[\frac{1}{2}(\lambda^2 X + Y^2)\right](t) = \mathbb{E}\left[\frac{1}{2}(\lambda^2 X_0^2 + Y_0^2)\right] + \frac{t}{2}$

à préserver

quels schémas numériques... ?