Fourier series and applications for the heat equation

Bachelor thesis

Nora Bemanian

Supervisors: Adrien Laurent & Frédéric Valet
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Contents

1 Introduction

In this project I will be giving an introduction to Fourier series, some theory about their convergence and some applications for solving PDE's, in my case the heat equation. The goal is to give an overview of the topic of Fourier series, as well as some applications. I have come across the topic Fourier series in many of the courses here at university, which is not surprising as I have learned that Fourier series have a lot of applications. The application that I will be focusing on in this thesis is for solving partial differential equations, namely the heat equation.

In the first section of the thesis, I introduce Fourier series, including the derivation of the coefficients. Following this introduction, I will discuss some convergence-theorems, and I will prove some completeness of Fourier series. Following this I will look at what happens to Fourier series that have jump discontinuities (Gibbs phenomenon). Lastly I will show how to use the theory on Fourier series for solving the inhomogeneous heat equation, which is a useful application as mentioned above. At the end of the thesis, there will be a short introduction to a numerical approach, called the method of finite differences for solving the heat equation.

My supervisors for this project are Adrien Ange André Laurent and Frédéric Fernand Jacques Valet.

2 Theoretical part

For the first part of this thesis, I will be introducing theory regarding Fourier series, and a theoretical approach to its applications.

2.1 Preliminaries

In this section, I introduce some definitions and formulas that I use in the thesis. All the definitions and theorems below are from the book "Partial differential equations - an introduction" by Walter A. Strauss, unless another refrence is specified.

Definition 2.1. *[2] Periodic function: A function f(x) is periodic with period p if* $f(x+p) = f(x)$.

Definition 2.2. *[2] Even and odd functions: A function f(x) is* ***even*** *if* $f(-x) = f(x)$, *and* ***odd*** *if* $f(-x) = -f(x)$. *An important example for us is that* $\cos n\theta$ *is an even function, and* $\sin n\theta$ *is an odd function.*

Facts about even and odd functions:

1. The integral of odd functions over symmetric intervals is zero.

2. The sum of two even functions is even and the sum of two odd functions is odd

3. The product of one even function and one odd function is odd

4. The product of two even functions or two odd functions is even

DeMoivre formulas

$$\begin{cases} \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} \\ \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} \end{cases} \tag{2.1}$$

Eulers formula:

$$e^{i\theta} = \cos(i\theta) + i\sin(i\theta) \tag{2.2}$$

Definition 2.3. *Inner product: The inner product of two complex valued functions f(x) and g(x) on the interval (a,b) is defined as*

$$(f,g) = \int_a^b f(x)\overline{g(x)}dx$$

where $\overline{g(x)}$ is the complex conjugate of g. In case of real valued functions f(x) and g(x):

$$(f,g) = \int_a^b f(x)g(x)dx$$

Definition 2.4. *[4] Orthogonal set of functions: Let $f_{n_{n\in\mathbb{Z}}}$ be a sequence of complex valued functions on [a,b] such that their inner product is zero:*

$$\int_a^b f_n(x)\overline{f_m(x)}dx = 0, \quad (n \neq m).$$

Then the sequence of functions is an orthogonal set of functions on the interval. If additionally $\int_a^b |f_n(x)|^2 dx = 1$ for all n, the sequence of functions is orthonormal.

2.2 What are Fourier series?

Fourier series are used to represent periodic functions as sums of cosine and sine functions. This works well because cosine and sine are periodic. For some function f(x) on an interval $-l < x < l$, the full real Fourier series is given by:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)) \tag{2.3}$$

The series consists of the Fourier cosine series, and the Fourier sine series, which are respectively even and odd. It is possible to rewrite (2.3) as the complex Fourier series by using the DeMoivre formulas (2.1) [2]:

$$\begin{aligned}
f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)) \\
&= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \frac{e^{\frac{in\pi x}{l}} + e^{\frac{-in\pi x}{l}}}{2} + b_n \frac{e^{\frac{in\pi x}{l}} - e^{\frac{-in\pi x}{l}}}{2i}\right) \\
&= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \frac{e^{\frac{in\pi x}{l}}}{2} + a_n \frac{e^{\frac{-in\pi x}{l}}}{2} - ib_n \frac{e^{\frac{in\pi x}{l}}}{2} + ib_n \frac{e^{\frac{-in\pi x}{l}}}{2}\right) \\
&= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\left(\frac{a_n - ib_n}{2}\right) e^{\frac{in\pi x}{l}} + (\left(\frac{a_n + ib_n}{2}\right) e^{\frac{-in\pi x}{l}}\right)
\end{aligned}$$

Now this is written in a complex form, and I will rename the coefficients by the following relations:

$$\begin{cases}
c_0 = \frac{1}{2}a_0 \\
c_n = \left(\frac{a_n - ib_n}{2}\right) \\
c_{-n} = \left(\frac{a_n + ib_n}{2}\right)
\end{cases}$$

which can be rewritten as:

$$\begin{cases}
a_0 = 2c_0 \\
a_n = c_n + c_{-n} \\
b_n = i(c_n - c_{-n})
\end{cases}$$

I can rewrite the complex form of the Fourier series by using these relations:

$$\begin{aligned}
f(x) &= \frac{1}{2}2c_0 + \sum_{n=1}^{\infty}\left(c_n e^{\frac{in\pi x}{l}} + c_{-n} e^{\frac{-in\pi x}{l}}\right) \\
&= \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}
\end{aligned}$$

So the **complex Fourier series** is:

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} \tag{2.4}$$

In the following sections, I will show how to derive the coefficients a_n, b_n and c_n of the Fourier series. In order to derive the Fourier coefficients it is necessary to familiarize the notion of orthogonlity.

2.2.1 Orthogonality

In order to derive the coefficients, I make use of the property of orthogonality, defined in 2.4. This property is particularly useful in the case of cosine and sine function to derive the coefficients of the real Fourier series. Here I can use the fact that

$$\int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx = 0 \quad \text{for} \quad m \neq n \tag{2.5}$$

Proof. The proof that I provide here is based on the book "Partial differential equations - an introduction" by Walter A Strauss. Using the trigonometric identity $\sin(a)\sin(b) = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$, the left hand side of (2.5) can be rewritten as:

$$\begin{aligned}
\int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx &= \int_{-l}^{l} \frac{1}{2}\cos\left(\frac{(n-m)\pi x}{l}\right) - \frac{1}{2}\cos\left(\frac{(n+m)\pi x}{l}\right) dx \\
&= \frac{1}{2}\int_{-l}^{l} \cos\left(\frac{(n-m)\pi x}{l}\right) dx - \frac{1}{2}\int_{-l}^{l} \cos\left(\frac{(n+m)\pi x}{l}\right) dx \\
&= \frac{1}{2}\left[\frac{l}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{l}\right)\right]_{-l}^{l} \\
&- \frac{1}{2}\left[\frac{l}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{l}\right)\right]_{-l}^{l}
\end{aligned}$$

Here there are three cases to consider; one where $m \neq n$, one where $m = n \neq 0$ and lastly the case where $m = n = 0$. For the first case, when $\mathbf{m} \neq \mathbf{n}$, all terms will disappear as I get sine of some integer multiplied by π in every case. For the second case, when $\mathbf{m} = \mathbf{n} \neq \mathbf{0}$, there will be an issue because of zero in the denominator. To avoid this, I go back to the stage of the integral where

$$\int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx = \frac{1}{2}\int_{-l}^{l} \cos\left(\frac{(n-m)\pi x}{l}\right) dx - \frac{1}{2}\int_{-l}^{l} \cos\left(\frac{(n+m)\pi x}{l}\right) dx$$

and I put in m=n:

$$\begin{aligned}
&= \frac{1}{2}\int_{-l}^{l} \cos(0) dx - \frac{1}{2}\int_{-l}^{l} \cos\left(\frac{2m\pi x}{l}\right) dx \\
&= \frac{1}{2}\int_{-l}^{l} dx - \frac{1}{2}\int_{-l}^{l} \cos\left(\frac{2m\pi x}{l}\right) dx \\
&= \frac{1}{2}[x]_{-l}^{l} - \frac{1}{2}\left[\frac{l}{2m\pi x}\sin\left(\frac{2m\pi x}{l}\right)\right]_{-l}^{l} \\
&= l
\end{aligned}$$

For the third case, when $\mathbf{m} = \mathbf{n} = \mathbf{0}$, $\int_{-l}^{l} \sin\left(\frac{n\pi 0}{l}\right)\sin\left(\frac{m\pi 0}{l}\right)dx = \int_{-l}^{l} 0 = 0$
To conclude, I have the following the property:

$$\int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = \begin{cases} 0 & \text{if} \quad m \neq n \quad \text{or} \quad m = n = 0 \\ l & \text{if} \quad m = n \end{cases} \tag{2.6}$$

□

Similarly, I have:

$$\int_{-l}^{l} \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ l & \text{if} \quad m = n \neq 0 \\ 2l & \text{if} \quad m = n = 0 \end{cases} \tag{2.7}$$

In this case, when m=n=0, we have $\int_{-l}^{l} \cos\left(\frac{n\pi 0}{l}\right)\cos\left(\frac{m\pi 0}{l}\right)dx = \int_{-l}^{l} 1dx = [x]_{-l}^{l} = 2l$ The rest of the proof is omitted since it is a lot like the previous proof.

The last orthogonality property that we will be using is:

$$\int_{-l}^{l} \cos\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right) = 0 \tag{2.8}$$

This is true because cosine is an even function, and sine is an odd function, and since an even function multiplied by an odd function is odd, I have the integral over -l to l (symmetric about 0) of an odd function. This is always 0.

2.2.2 The coefficients

Now I will be using property (2.6), (2.7) and (2.8) to find the coefficients of (2.3).

Finding a_0:
To find the coefficient a_0, I take the integral on both sides of (2.3):

$$
\begin{aligned}
\int_{-l}^{l} f(x)dx &= \int_{-l}^{l} \frac{1}{2}a_0 + \int_{-l}^{l} \sum_{n=1}^{\infty}(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)) \\
&= \frac{1}{2}a_0 \int_{-l}^{l} 1dx + \sum_{n=1}^{\infty} \int_{-l}^{l} a_n \cos\left(\frac{n\pi x}{l}\right)dx + \sum_{n=1}^{\infty} \int_{-l}^{l} b_n \sin\left(\frac{n\pi x}{l}\right))dx \\
&= \frac{1}{2}a_0 \int_{-l}^{l} 1dx \\
&= a_0 l
\end{aligned}
$$

Thus, a_0 is given by:

$$
\begin{aligned}
\int_{-l}^{l} f(x)dx &= a_0 l \\
a_0 &= \frac{1}{l}\int_{-l}^{l} f(x)dx
\end{aligned}
\tag{2.9}
$$

Finding a_n:
For finding a_n, I multiply both sides of (2.3) by $\cos\left(\frac{m\pi x}{l}\right)$ and integrate on both sides of the equal sign so that I can use the orthogonality properties (2.7) and (2.8):

$$
\begin{aligned}
\int_{-l}^{l} f(x)\cos\left(\frac{m\pi x}{l}\right)dx = \frac{1}{2}a_0 \int_{-l}^{l} \cos\left(\frac{m\pi x}{l}\right) + \sum_{n=1}^{\infty}\int_{-l}^{l} a_n \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx \\
+ \sum_{n=1}^{\infty}\int_{-l}^{l} b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx
\end{aligned}
$$

The first term on the right hand side,

$$
\frac{1}{2}a_0 \int_{-l}^{l} \cos\left(\frac{m\pi x}{l}\right)
$$

.....her må det stå noe. The last term,

$$
\sum_{n=1}^{\infty}\int_{-l}^{l} a_n \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx
$$

, is also 0, by (2.8). The second term,$\sum_{n=1}^{\infty}\int_{-l}^{l} a_n \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx$, is 0 whenever $m \neq n$. For the cases where m=n I have to consider m $= 0$ and $m \neq 0$.

For $m = 0$, I get:

$$\int_{-l}^{l} a_n \cos\left(\frac{n\pi x}{l}\right) \cos\left(\frac{m\pi x}{l}\right) dx = \int_{-l}^{l} a_n \cos\left(\frac{n\pi x}{l}\right) dx$$

which is 0.

For the case where $m = n \neq 0$, I get:

$$\begin{aligned}
\int_{-l}^{l} a_n \cos\left(\frac{n\pi x}{l}\right) \cos\left(\frac{m\pi x}{l}\right) dx &= \int_{-l}^{l} a_n \cos^2\left(\frac{n\pi x}{l}\right) dx \\
&= a_n \int_{-l}^{l} \frac{1}{2}\left[\cos\left(\frac{2n\pi x}{l}\right) + 1\right] dx \\
&= \frac{a_n}{2}\left[\int_{-l}^{l} \cos\left(\frac{2n\pi x}{l}\right) dx + \int_{-l}^{l} 1 dx\right] \\
&= \frac{a_n}{2}\left[\left[\frac{l}{2n\pi} \sin\left(\frac{2n\pi x}{l}\right)\right]_{-l}^{l} + [x]_{-l}^{l}\right] \\
&= \frac{a_n}{2}[0 + 2l] \\
&= a_n l
\end{aligned}$$

So I have that all terms on the left side disappear, except $a_n l$, which gives me the expression for a_n:

$$\begin{aligned}
\int_{-l}^{l} f(x) \cos\frac{m\pi x}{l} dx &= a_n l \\
a_n &= \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{m\pi x}{l} dx
\end{aligned} \tag{2.10}$$

Finding b_n: To derive b_n I multiply (2.3) by $\sin\left(\frac{m\pi x}{l}\right)$ and integrate on both sides of =:

$$\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx = \frac{1}{2}a_0\int_{-l}^{l}\sin\left(\frac{m\pi x}{l}\right)dx + \sum_{n=1}^{\infty} a_n\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx$$

$$+\sum_{n=1}^{\infty} b_n\int_{-l}^{l}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx$$

$$= 0 + 0 + b_n l$$

(By the orthogonality properties). So I am left with

$$\begin{aligned}\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx &= b_n l \\ b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx\end{aligned} \tag{2.11}$$

2.3 Complex form of the Fourier series

The complex form of the Fourier series is given by:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

I recall that $c_n = \frac{a_n - ib_n}{2}$ from the relations in section 2.2. Now that I know what a_n and b_n are, I can put them into the equation:

$$\begin{aligned}c_n &= \frac{a_n - ib_n}{2} \\ &= \frac{\frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{m\pi x}{l}\right)dx - i\frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx}{2} \\ &= \frac{1}{2l}\int_{-l}^{l} f(x)\left[\cos\left(\frac{m\pi x}{l}\right) - i\sin\left(\frac{m\pi x}{l}\right)\right]dx \\ &= \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{im\pi x}{l}}dx\end{aligned}$$

So the coefficient for the complex Fourier expansion is:

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{-in\pi x}{l}} \tag{2.12}$$

2.4 Completeness

In order to discuss the completeness of the Fourier series, I must introduce some theorems. All of the theorems and definitions below are from the book "Partial differential equations - an introduction" by Walter A. Strauss, with some changes on the notation.

2.4.1 Convergence theorems

Definition 2.5 (Pointwise convergence). *An infinite series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise in (a,b) if for each $a < x < b$,*

$$|f(x) - \sum_{n=1}^{N} f_n(x)| \underset{N\to\infty}{\longrightarrow} 0$$

Definition 2.6 (Uniform convergence). *An infinite series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to f(x) in [a,b] if*

$$\underset{a\leqslant x\leqslant b}{max}|f(x) - \sum_{n=1}^{N} f_n(x)| \underset{N\to\infty}{\longrightarrow} 0$$

Definition 2.7 (L^2-convergence). *An infinite series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges in the L^2 sense to f(x) in (a,b) if*

$$\int_a^b |f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \underset{N\to\infty}{\longrightarrow} 0$$

Theorem 2.8. *Uniform convergence of Fourier series: The Fourier series of f(x) on [a,b] converges uniformly to f(x) if*

1. *The function, as well as its first and second derivatives exist and are continuous on [a,b]*

2. *The function satisfies the boundary conditions.*

Theorem 2.9. *L^2 convergence of Fourier series: The Fourier series converges to f(x) in the sense of L^2 on (a,b) if $\int_a^b |f(x)|^2 dx$ is finite.*

Theorem 2.10. *Pointwise convergence of Fourier series: The Fourier series converges pointwise to f(x) on the interval (a,b) if f(x) is continuous and f'(x) is piecewise continuous for* $x \in [a,b]$*. If f(x) is only piecewise continuous and f'(x) is piecewise continuous as before, and f(x) is 2l-periodic, then the Fourier series converges to* $\frac{1}{2}[f(x+) + f(x-)]$ *at every point* $x \in (-\infty, \infty)$*.*

Theorem 2.11. *If* $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly to f(x), then* $\sum_{n=1}^{\infty} f_n(x)$ *also converges in the* L^2*-sense to f(x).*

Proof. I let the series $\sum_{n=1}^{\infty} f_n(x)$ be uniformly convergent in [a,b], so

$$\underset{a\leqslant x\leqslant b}{max}|f(x) - \sum_{n=1}^{N} f_n(x)| \underset{N\to\infty}{\longrightarrow} 0,$$

or in other words:

$$\underset{N\to\infty}{lim}\,\underset{a\leqslant x\leqslant b}{max}|f(x) - \sum_{n=1}^{N} f_n(x)| = 0.$$

I have that:

$$\begin{aligned}
0 &\leqslant |f(x) - \sum_{n=1}^{N} f_n(x)| \leqslant \underset{a\leqslant x\leqslant b}{max}|f(x) - \sum_{n=1}^{N} f_n(x)| \\
0^2 &\leqslant |f(x) - \sum_{n=1}^{N} f_n(x)|^2 \leqslant \underset{a\leqslant x\leqslant b}{max}|f(x) - \sum_{n=1}^{N} f_n(x)|^2 \\
\int_a^b 0dx &\leqslant \int_a^b |f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \leqslant \int_a^b \underset{a\leqslant x\leqslant b}{max}|f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \\
\underset{N\to\infty}{lim} 0 &\leqslant \underset{N\to\infty}{lim}\int_a^b |f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \leqslant \underset{N\to\infty}{lim}\int_a^b \underset{a\leqslant x\leqslant b}{max}|f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \\
0 &\leqslant \underset{N\to\infty}{lim}\int_a^b |f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \leqslant \int_a^b \underset{N\to\infty}{lim}\,\underset{a\leqslant x\leqslant b}{max}|f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \\
0 &\leqslant \underset{N\to\infty}{lim}\int_a^b |f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \leqslant \int_a^b 0dx
\end{aligned}$$

With the assumption that $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent in [a,b], I have by the dominated convergence theorem theorem that the limit and the integral can be interchanged, as in the fifth line. Since I assumed uniform convergence, I have that the right hand side of the inequality goes to zero, which means that

$$0 \leqslant \int_a^b |f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \leqslant 0$$

Which means that $\int_a^b |f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \overset{N\to\infty}{\to} 0$ which is the definition of L^2-convergence.

□

2.5 Alternative proof of pointwise convergence

In this section, I will be giving an alternative proof for pointwise convergence of Fourier series. The proof is based on the proof of pointwise convergence published in American Mathematical Monthly by Paul R. Chernoff. I will be following the structure of exercise 13 of section 5.5 in the book "Partial differential equations - an introduction" by Walter A. Strauss, with the help of the proof by Chernoff [1]. Since I will be assuming that the function in question is C^1, this is a weaker proof than the usual proof, which can be found in the appendix, A. The proof of Chernoff is extended to deal with jump discontinuities as well, but I will omit this.

Proof. As mentioned, I start by letting $f(x)$ be a C^1 function with period 2π. I first want to show that I may assume f(0)=0. For this, I construct a function $h(x) = f(x) - f(0)$. Since f(x) is 2π-periodic, I have that:

$$h(x + 2\pi) = f(x + 2\pi) - f(0) = f(x) - f(0) = h(x),$$

so h(x) is 2π-periodic and $h(0) = 0$. The function f(x) shares the same properties as h(x), so I may assume that f(0) = 0.

Next up is to show that $g(x) = \frac{f(x)}{e^{ix}-1}$ is a continuous function. The numerator is just f(x) which I already assume is C^1, so discontinuities may appear when the denominator is zero:

$$\begin{aligned} e^{ix} - 1 &= 0 \\ e^{ix} &= 1 \\ cos(x) + isin(x) &= 1 \end{aligned}$$

which is the case whenever $x = 2n\pi$. Since sine is 2π-periodic, it is enough to show continuity when x=0 as it will attain the value 0 for each n=1,2,3.... To show continuity, I need to show that $lim_{x\to 0}g(x) = g(0)$

$$\begin{aligned} lim_{n\to 0}g(x) &= \frac{0}{0} \\ &\text{(L'hôpital)} \\ &= lim_{n\to 0}\frac{f'(x)}{ie^{ix}} \\ &= \frac{f'(x)}{i} \end{aligned}$$

Knowing that f(x) is C^1, so f'(x) is continuous , and i is the imaginary unit which is non-zero, I can conclude that g(x) is a continuous function.

Letting C_n be the complex Fourier coefficient of f(x) and D_n the complex coefficient of g(x). In part b, I concluded that g is everywhere continuous, and I showed that it is bounded when the denominator goes to 0. Because g(x) is finite, $\int_{-\pi}^{\pi} |g(x)|^2 dx$ is also finite. I can use Bessels inequality to state that:

$$\sum_{n=1}^{\infty} D_n^2 \int_{-\pi}^{\pi} |e^{-inx}|^2 \leqslant \int_{-\pi}^{\pi} |g(x)|^2$$

which is finite so is less than ∞. Now if $n \to \infty$, $D_n \to 0$ for this inequality to hold (because the right hand side of the inequality is finite).

Next, I will show that $C_n = D_{n-1} - D_N$ so that the series $\sum C_n$ is a telescoping series. I recall that a telescoping series is a series such that the terms a_n can be written as $b_n - b_{n+1}$. The partial sum of a telescoping series consists only of the first and the last term, as every other term cancels out.(ref) In this case, we have $f(x) = g(x)(e^{ix} - 2)$. Fourier expanding:

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)e^{-in\pi}dx &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)(e^{ix} - 1)e^{-inx}dx \\ \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)e^{-in\pi}dx &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)(e^{ix(1-n)} - \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}dx \\ C_n = D_{n-1} - D_n \sum_{n=-M}^{N} C_n &= \sum_{n=-M}^{N} D_{n-1} - D_n \end{aligned}$$

Because $D_n \to 0$, and since we have proved that the Fourier coefficient of f(x), C_n, can be written as a telescoping series of $D_{n-1} - D_n$ we know that C_n is 0 at x = 0, and thus the Fourier series of f(x) at x=0 converges to 0. Hence I have proved the pointwise convergence of the Fourier series for a C^1 function. $\square$

2.6 Gibbs phenomenon

Given a function f(x) that has jump discontinuities, the partial sum S_N is defined to approximate the jumps. It turns out by Gibbs phenomenon, that S_N differs from the function by around 9% ("overshoot"). The width of the jump goes to zero whenever N goes to infinity, but the height remains at 9%. In other words,

$$lim_{N\to\infty} max|S_N(x) - f(x)| \neq 0,$$

even though $lim_{N\to\infty}|S_N(x) - f(x)| \xrightarrow{N\to\infty} 0$ whenever there is not a jump[5]. One example of a function with a jump, from [5] is the following: $f(x) = \frac{1}{2}$ for $0 < x < \pi$, and $f(x) = \frac{-1}{2}$ for $-\pi < x < 0$ The Fourier sine series for this function, which is odd, is:

$$\sum_{n=1}^{\infty} \frac{2}{n\pi} sin(n\pi).$$

Using the Dirichlet Kernel from previously, the partial sums are:

$$S_N(x) = (\int_0^{\pi} - \int_{-pi}^{0}) K_N(x-y)\frac{dy}{4\pi} = (\int_0^{\pi} - \int_{-\pi}^{0})\frac{sin[(N+\frac{1}{2}(x-y)]}{sin[\frac{1}{2}(x-y)]}\frac{dy}{4\pi}$$

Defining $M = N + \frac{1}{2}$ and using change of variables $\theta = M(x-y)$ in the first integral and $\theta = M(y-x)$ in the second.

$$S_N(x) = (\int_{M(x-\pi)}^{Mx} - \int_{-M(x+\pi)}^{-Mx})\frac{sin(\theta)}{2M sin(\frac{\theta}{2M})}\frac{d\theta}{2\pi}$$
$$= (\int_{-Mx}^{Mx} - \int_{-M\pi-Mx}^{-M\pi+Mx})\frac{sin(\theta)}{2M sin(\frac{\theta}{2M})}\frac{d\theta}{2\pi}$$

Since the integral is even, I have:

$$= (\int_{-Mx}^{Mx} - \int_{M\pi-Mx}^{M\pi+Mx})\frac{sin(\theta)}{2M sin(\frac{\theta}{2M})}\frac{d\theta}{2\pi}$$

The max of the first integral here, is given by setting its derivative equal to zero. This is the case whenever sin((Mx) = 0, thus $x = \pi$. Then I have:

$$S_N(\frac{\pi}{M}) = (\int_{-\pi}^{\pi} - \int_{M\pi-\pi}^{M\pi+\pi})\frac{sin(\theta)}{2M sin(\frac{\theta}{2M})}\frac{d\theta}{2\pi}$$

For M>2: $\frac{\pi}{4} < [1 - \frac{1}{M}]\frac{\pi}{2} \leqslant \frac{\theta}{2M} \leqslant [1 + \frac{1}{M}]\frac{\pi}{2} < \frac{3\pi}{4}$ So $sin(\frac{\theta}{2M}) > \frac{1}{\sqrt{2}}$, which means the second integral is less than $\int_{M\pi-\pi}^{M\pi+\pi} 1 \cdot [\frac{2M}{\sqrt{2}}]^2 \frac{d\theta}{2\pi} = \frac{1}{\sqrt{2}M}$ which goes to

0 when M goes to infinity. The first integral, $|\theta| \leqslant \pi$ and $2Msin(\frac{\theta}{2M}) \xrightarrow{M\to\infty} \theta$ uniformly in $|\theta| \leqslant \pi$. Taking the limit of S_n as M goes to infinity gives me: $S_N(\frac{\pi}{M} \xrightarrow{M\to\infty} \int_{-\pi}^{pi} \frac{sin(\theta)}{\theta}\frac{d\theta}{2\pi} \simeq 0,59$ which is the Gibbs 9 percent overshoot. This presentation of the Gibbs phenomenon, as well as the example is entirely based section 5.5 of the book of Strauss mentioned earlier.

2.7 Application of Fourier series for solving a partial differential equation

In this section, I will be showing how Fourier series can be applied to solve partial differential equations. Later on, in section 3, I will solve the heat equation numerically. In this section will specifically be solving the heat equation with inhomogeneous boundary conditions and a source term. Physically, the heat equation in one dimension paired with its boundary conditions and initial condition describes the rate that the temperature changes along a rod. The inhomogeneous heat equation with a source term and Dirichlet boundary conditions is given by the following:

$$\begin{cases} u_t - u_{xx} = f(x,t) \\ u(0,t) = 0, \quad u(l,t) = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{2.13}$$

Where f(x,t) is the source term. Fourier expanding u(x,t) by the Fourier sine series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) sin(\frac{n\pi x}{l})$$

where

$$u_n = \frac{2}{l}\int_0^l u(x,t) sin(\frac{n\pi x}{l})dx$$

Letting

$$u_t = \sum_{n=1}^{\infty} v_n(t) sin(\frac{n\pi x}{l})$$

where

$$v_n = \frac{2}{l}\int_0^l \frac{\partial u}{\partial t} sin(\frac{n\pi x}{l})dx$$

also letting

$$u_{xx} = \sum_{n=1}^{\infty} w_n(t) sin(\frac{n\pi x}{l})$$

where

$$w_n = \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial t^2} sin(\frac{n\pi x}{l})dx$$

I want to use Greens second identity:

$$\int_a^b (-X_1''X_2 + X_1X_2'')dx = (-X_1'X_2 + X_1X_2')|_a^b$$

I make

$$X_1 = sin(\frac{n\pi x}{l}), \quad X_1' = (\frac{n\pi}{l})cos(\frac{n\pi x}{l}), \quad X_1'' = -(\frac{n\pi}{l})^2 sin(\frac{n\pi x}{l})$$

and

$$X_2 = u, \quad X_2' = u_x, \quad X_2'' = u_x x$$

Putting this into Greens second identity:

$$\int_0^l \left(\frac{n\pi}{l}\right)^2 \sin\left(\frac{n\pi x}{l}\right) u dx - \int_0^l sin\Big(\frac{n\pi x}{l}\Big) u_{xx} dx = \Big[-\left(\frac{n\pi}{l}\right)\cos\left(\frac{n\pi x}{l}\right)u + sin\Big(\frac{n\pi x}{l}\Big)u_x\Big]_0^l$$

Note that the last term on the left hand side $-\int_0^l sin\big(\frac{n\pi x}{l}\big)u_{xx}dx = -\frac{l}{2}w_n(t)$. In other words, we now have:

$$\begin{aligned}
w_n(t) &= -\frac{2}{l}\left[\int_0^l \left(\frac{n\pi}{l}\right)^2 \sin\left(\frac{n\pi x}{l}\right) u dx + \Big[-\left(\frac{n\pi}{l}\right)\cos\left(\frac{n\pi x}{l}\right)u + sin\Big(\frac{n\pi x}{l}\Big)u_x\Big]_0^l\right] \\
&= -\frac{2}{l}\left(\frac{n\pi}{l}\right)^2 \int_0^l \sin\left(\frac{n\pi x}{l}\right) u dx - \frac{2}{l}\Big[-\left(\frac{n\pi}{l}\right)\cos\left(\frac{n\pi x}{l}\right)u + sin\Big(\frac{n\pi x}{l}\Big)u_x\Big]_0^l \\
&= -\frac{2}{l}\int_0^l \left(\frac{n\pi}{l}\right)^2 u_n(t)
\end{aligned}$$

The last term disappears because of the boundary conditions. I am now left with only

$$w_n(t) = -\frac{2}{l}\int_0^l \left(\frac{n\pi}{l}\right)^2 u_n(t)$$

The PDE is $u_t - u_x x = f(x,t)$, which in terms of the coefficients means that we require:

$$v_n(t) - w_n(t) = \frac{2}{l}\int_0^l u_t sin(\frac{n\pi x}{l}) + \left(\frac{n\pi}{l}\right)^2 u_n(t)dx = f_n(t)$$

This way the partial differential equation can be rewritten into an ordinary differential equation, and may be solved given the source term and initial condition.

3 The method of finite differences for heat equation

In the previous section, 2.7, I solved the heat equation with a source term by using Fourier expansion. In this section, I will show a numerical method for approximately solving the heat equation, with and without a source term. Partial differential equations, such as the heat equation involve derivatives, which I will approximate numerically using the difference formulas. From the definition of the derivative, $f'(x) = lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, and for a sufficiently small (non-zero) h, the derivative, $f'(x)$ can be approximated by $f'(x) \approx \frac{f(x+h)-f(x)}{h}$. This is the forward difference. The backward difference is $f'(x) \approx \frac{f(x)-f(x-h)}{h}$. The central difference is really just the mean of these the forward difference and the backward difference: $f'(x) \approx \frac{f(x+h)-f(x)+f(x)-f(x-h)}{2h} = \frac{f(x+h)-f(x-h)}{2h}$. The central difference for the second derivative is $f''(x) \approx \frac{f(x+h)-2f(x-h)}{2h}$. The finite difference formula, as well as what follows below in this section is based on the book "A First Course in the Numerical Analysis of Differential Equations" by Arieh Iserles [3].

3.1 The finite difference scheme for the heat equation without source term

The heat equation, here with Dirichlet boundary conditions and without source term, given by the following

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = 0, \quad u(l,t) = 0 \\ u(x,0) = u_0(x), \end{cases} \tag{3.1}$$

is a PDE that depends on time, t, and space, x. A PDE that depends on both space and time is called an evolutionary equation [3]. However, in order to use the finite difference formulas from above it must be dependent of only one variable. I discretize x by defining the step size along the x-axis to be $\Delta x = \frac{L}{d+1}$, where L is the length of the rod and $d \in \mathbb{Z}^+$. Each $x_\ell = \ell\Delta x$ denotes a grid point along x. The step size of the time is $\Delta t = \frac{T}{M}$ where $M \in \mathbb{Z}^+$. Each node in the grid provides a solution at some time t. The central difference method for approximating second derivatives, as presented above, now yields:

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{u(x-\Delta x,t) - 2u(x,t) + u(x+\Delta x,t)}{(\Delta x)^2} + \mathcal{O}((\Delta x)^2)$$

where $\mathcal{O}(\Delta x)^2$ denotes the error. I use the forward difference formula for the first derivative of u with respect to t:

$$\frac{\partial u(x,t)}{\partial t} \approx \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} + \mathcal{O}(\Delta t).$$

Putting these back into the PDE (3.1):

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} + \mathcal{O}(\Delta t) = \frac{u(x-\Delta x,t) - 2u(x,t) + u(x+\Delta x,t)}{(\Delta x)^2}\mathcal{O}((\Delta x)^2).$$

Multiplying on both sides with Δt and then moving $-u(x,t)$ and the error to the right hand side I get:

$$u(x,t+\Delta t) = u(x,t) + \Delta t\frac{u(x-\Delta x,t) - 2u(x,t) + u(x+\Delta x,t)}{(\Delta x)^2} + \mathcal{O}(\Delta t) - \mathcal{O}((\Delta x)^2) \tag{3.2}$$

I denote the approximated solution $u(\ell\Delta x, n\Delta t)$ by u_ℓ^n as in [3], so (3.2) becomes:

$$u_\ell^{n+1} \approx u_\ell^n + \frac{\Delta t}{(\Delta x)^2}(u_{\ell-1}^n - 2u_\ell^n + u_{\ell+1}^n), \quad \ell = 1, 2, ..., N, \quad n = 0, 1, ..., \tag{3.3}$$

Which is the Euler discretization in time. This can all be written in matrix form. I denote the vector form of the solution $U^n = (u_\ell^n)_\ell \in \mathbb{R}^d$, and rewrite (3.3) as the folllowing:

$$U^{n+1} = U^n + \frac{\Delta t}{(\Delta x)^2}AU^n, \quad n = 0, 1, ..., \tag{3.4}$$

Where A is a matrix with entries of -2 in the main diagonal, and 1 in the diagonals above and below. By the boundary conditions I have 0 everywhere else (left blank):

$$A = \begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}$$

A numerical example to illustrate the heat distribution:

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = 0, \quad u(l,t) = 0 \\ u(x,0) = \cos(2\pi x), \end{cases} \tag{3.5}$$

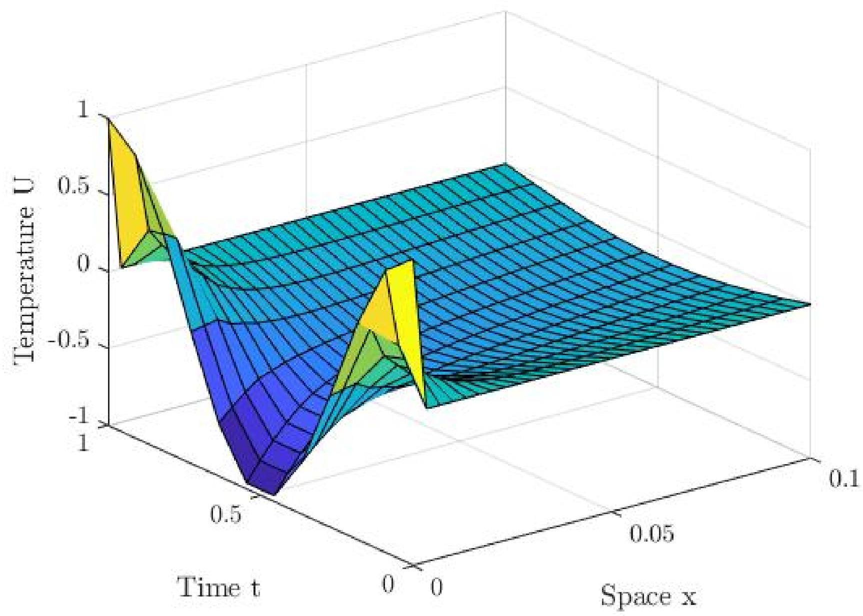

Figure 1: The finite difference approximation for the heat equation with Dirichlet boundary conditions, initial condition $u_0 = \cos 2\pi x$, and no source term

3.2 Finite difference method for the inhomogeneous heat equation with source term

The heat equation with a source term (again with Dircihlet boundary conditions):

$$\begin{cases} u_t - u_{xx} = f(u) \\ u(0,t) = 0, \quad u(l,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

The method of finite differences in this case gives me:

$$u_\ell^{n+1} = u_\ell^n + \frac{\Delta t}{(\Delta x)^2}(u_{\ell-1}^n - 2u_\ell^n + u_{\ell+1}^n) + \Delta t f(u_\ell^n), \quad \ell = 1, 2, ..., N, \quad n = 0, 1, ...$$

As previously in matrix form:

$$U^{(n+1)} = U^n + \frac{\Delta t}{(\Delta x)^2} AU^n + \Delta t F(U^n), \quad n = 0, 1, ..., \tag{3.6}$$

Where A is the same as previously and

$$F(U^n) = \begin{pmatrix} f(u_1^n) \\ \vdots \\ f(u_N^n) \end{pmatrix}$$

3.3 Bonus: Fourier method

One very last thing that is worth mentioning, is the Fourier method for solving partial differential equations. I unfortunately did not have the capacity to cover this in this thesis, but since I have already worked on a program for this method with my supervisor, I will include the result of the numerical example from 3.5 solved using the Fast Fourier Transform algorithm, as a bonus:

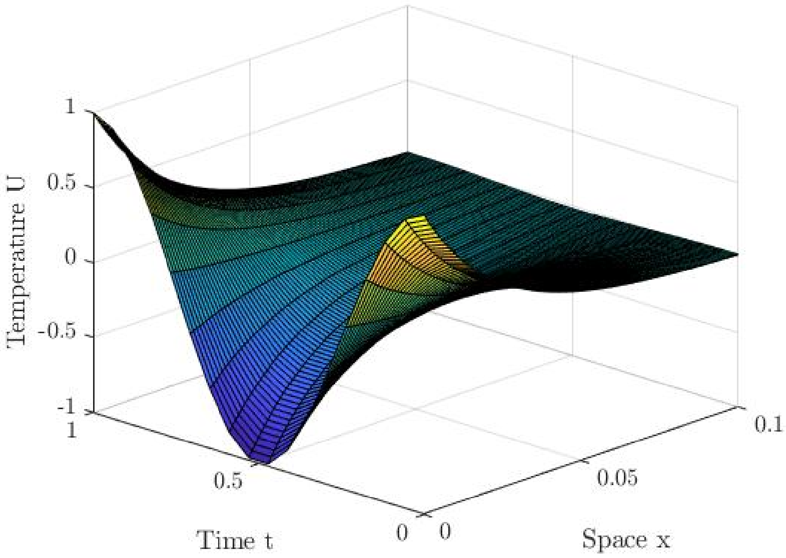

Figure 2: The finite difference approximation for the heat equation with Dirichlet boundary conditions, initial condition $u_0 = \cos 2\pi x$, with source term and using FFT

References

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[4] W. Rudin et al. *Principles of mathematical analysis*, volume 3. McGraw-hill New York, 1976.

[5] W. A. Strauss. *Partial differential equations: An introduction*. John Wiley & Sons, 2007.

Appendices

A Proof of pointwise convergence convergence of Fourier series

I will prove the theorem for pointwise convergence of Fourier series 2.10, as in section 5.5 of "Partial Differential Equations - an introduction" by Walter A. Strauss [5]. In the proof I will be making use of Bessels inequality, which states that for an orthogonal set of real valued functions $\{f_n\}_{n\in\mathbf{Z}}$, and a function g(x), for which the L^2-norm: $||g(x)|| = \left[\int_a^b |g(x)|^2 dx\right]^{\frac{1}{2}}$ is finite, we have the following inequality:

$$\sum_{n=1}^{\infty} c_n^2 \int_a^b |f_n(x)|^2 dx \leqslant \int_a^b |g(x)|^2 dx \tag{A.1}$$

where c_n is the Fourier coefficient. To avoid creating introducing more new concepts and notation, I will not provide the derivation of this inequality.

Proof. For the proof, I will first consider C^1 function of period 2π and then see what happens when there are jump discontinuities. In the case of a C^1-function of period of 2π, such that $l = \pi$, the full Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos(nx) + b_n sin(nx)),$$

where

$$a_n = \int_{-\pi}^{\pi} f(y)cos(ny)\frac{1}{\pi}dy \quad (n = 0, 1, 2, ...)$$

, and

$$n_n = \int_{-\pi}^{\pi} f(y)sin(ny)\frac{1}{\pi}dy \quad (n = 1, 2, ...).$$

I define the Nth partial sum of the Fourier series as

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n cos(nx) + b_n sin(nx)).$$

In order to show pointwise convergence, I need to show that $|f(x) - S_N| \underset{N\to\infty}{\longrightarrow} 0$. I rewrite the expression for the partial sum by replacing a_n and b_n by their respective expressions. Here, I note that $a_0 = \int_{-\pi}^{\pi} f(y)cos(0)\frac{dy}{\pi} = \int_{-\pi}^{\pi} f(y)\frac{dy}{\pi}$ so I have:

$$S_N(x) = \frac{1}{2}\int_{-\pi}^{\pi} f(y)\frac{1}{\pi}dy + \sum_{n=1}^{N}\int_{-\pi}^{\pi} f(y)cos(ny)cos(nx)\frac{1}{\pi}dy + \sum_{n=1}^{N}\int_{-\pi}^{\pi} f(y)sin(ny)sin(nx)\frac{1}{\pi}dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1 + 2\sum_{n=1}^{N} cos(ny)cos(nx) + sin(ny)sin(nx)\right] f(y)dy.$$

From the trigonometric identity that $cos(a-b) = cos(a)cos(b) + sin(a)sin(b)$, I can rewrite the expression as:

$$S_N(x) = \int_{-\pi}^{\pi} [1 + 2\sum_{n=1}^{N}(cos(ny - nx)]\frac{f(y)}{\pi}dy$$

. The part $1 + 2\sum_{n=1}^{N}(cos(ny - nx)$ is called the Dirichlet kernel:

$$K_N(\theta) = 1 + 2\sum_{n=1}^{N} cos(n\theta) \tag{A.2}$$

In my case, I have $\theta = y - x$. The Dirichlet kernel, $K_N(\theta)$, can be rewritten in the form $K_N(\theta) = \frac{\sin\left[\left(N+\frac{1}{2}\right)\theta\right]}{\sin\left(\frac{1}{2}\theta\right)}$ by using De Moirves formulas (2.1) to replace $cos(n\theta)$ by $\frac{e^{in\theta}+e^{-in\theta}}{2}$:

$$\begin{aligned} K_N(\theta) &= 1 + 2\sum_{n=1}^{N}\frac{e^{in\theta} + e^{-in\theta}}{2} \\ &= 1 + \sum_{n=1}^{N} e^{in\theta} + e^{-in\theta} \\ &= \sum_{n=-N}^{N} e^{in\theta} \end{aligned}$$

I note that $\sum_{n=-N}^{N} e^{in\theta}$ is actually a finite geometric series, with ratio $e^{i\theta}$.

$$K_N(\theta) = \sum_{n=-N}^{N} e^{in\theta}$$

$$
\begin{aligned}
&= \sum_{k=0}^{2N} e^{i(k-N)\theta} \\
&= e^{-iN\theta} \sum_{k=0}^{2N} (e^{i\theta})^k \\
&= e^{-iN\theta} \left[\frac{1-(e^{i\theta})^{2N+1}}{1-e^{i\theta}} \right] \\
&= e^{-iN\theta} \left[\frac{1-(e^{i\theta})^{2N+1}}{1-e^{i\theta}} \right] \frac{e^{\frac{-i\theta}{2}}}{e^{\frac{-i\theta}{2}}} \\
&= e^{-iN\theta} \left[\frac{e^{-\frac{i\theta}{2}} - e^{2i\theta N+\frac{i\theta}{2}}}{e^{\frac{-i\theta}{2}} - e^{\frac{i\theta}{2}}} \right] \\
&= e^{-iN\theta} \left[\frac{e^{2i\theta N+\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}} - e^{\frac{-i\theta}{2}}} \right] \\
&= \frac{e^{i\theta N+\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}-i\theta N}}{e^{\frac{i\theta}{2}} - e^{\frac{-i\theta}{2}}}
\end{aligned}
$$

At this point, I note that by manipulation of the De Moivre formulas: (2.1):

$$
\begin{aligned}
\sin\frac{1}{2}\theta &= \frac{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}}{2i} \\
2i\sin\frac{1}{2}\theta &= e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}} \\
&= e^{\frac{i\theta}{2}}(1-e^{-i\theta})
\end{aligned}
$$

which is exactly the denominator.

$$
\begin{aligned}
K_N(\theta) &= \frac{e^{i\theta N+\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}-i\theta N}}{e^{\frac{i\theta}{2}} - e^{\frac{-i\theta}{2}}} \\
&= \frac{e^{i\theta N+\frac{i\theta}{2}} - e^{\frac{-i\theta}{2}-i\theta N}}{2i\sin\left(\frac{1}{2}\theta\right)}
\end{aligned}
$$

The nominator can be rewritten as $e^{i\theta(N+\frac{1}{2})} - e^{-i\theta(N+\frac{1}{2})}$, and I have:

$$
\begin{aligned}
K_N(\theta) &= \frac{e^{i\theta\left(N+\frac{1}{2}\right)} - e^{-i\theta\left(N+\frac{1}{2}\right)}}{2i} \frac{1}{\sin\left(\frac{1}{2}\theta\right)} \\
&= \sin{(N+\frac{1}{2})} \frac{1}{\sin\left(\frac{1}{2}\theta\right)} \\
&= \frac{\sin\left(N+\frac{1}{2}\right)}{\sin\left(\frac{1}{2}\theta\right)}
\end{aligned}
$$

As just shown, the Dirichlet kernel (A.2), can be written as:

$$
K_N(\theta) = \frac{\sin\left(N+\frac{1}{2}\right)}{\sin\left(\frac{1}{2}\theta\right)} \tag{A.3}
$$

Going back to the $S_N(x)$, where $\theta = y - x$, I have:

$$
\begin{aligned}
S_N(x) &= \int_{x-\pi}^{x+\pi} K_n(-\theta) f(\theta+x) \frac{d\theta}{2\pi} \\
&= \int_{-\pi}^{\pi} K_n(-\theta) f(\theta+x) \frac{d\theta}{2\pi}
\end{aligned}
$$

The interval of integration is rewritten as $[-\pi, \pi]$ because of the periodicity of K_n and f (2π-periodic). Because the Dirichlet kernel is the product of two odd functions, it is even (see preliminaries) and this is really the same as:

$$
\begin{aligned}
S_N(x) &= \int_{-\pi}^{\pi} K_n(\theta) f(\theta+x) \frac{d\theta}{2\pi} \\
&= \int_{-\pi}^{\pi} \frac{\sin\left(N+\frac{1}{2}\right)}{\sin\left(\frac{1}{2}\theta\right)} f(\theta+x) \frac{d\theta}{2\pi}
\end{aligned}
$$

Subtracting f(x) from both sides of the equation (f(x) is independent of θ so it is a constant in this case):

$$
S_N(x) - f(x) = \int_{-\pi}^{\pi} \frac{\sin\left(N+\frac{1}{2}\right)}{\sin\left(\frac{1}{2}\theta\right)} [f(\theta+x) - f(x)] \frac{d\theta}{2\pi}
$$

I let $g(\theta) = \frac{f(x+\theta)-f(x)}{sin(\frac{1}{2}\theta)}$ and replace accordingly:

$$
S_N(x) - f(x) = \int_{-\pi}^{\pi} g(\theta) sin[(N+\frac{1}{2})\theta] \frac{d\theta}{2\pi}
$$

Defining the set of functions $\{f_N(\theta)\}_{N\in\mathbf{Z}} = sin[(N+\frac{1}{2})\theta]$, which forms an orthogonal set. Thus we can use the fact that Bessels inequality (A.1) holds, so:

$$\sum_{N=1}^{\infty} \frac{|g, f_N|^2}{||f_N||^2} \leqslant ||g||^2$$

Calculating $||f_N||^2$:

$$\begin{aligned}
||f_N||^2 &= ||\sin\left[\left(N+\frac{1}{2}\right)\theta\right]||^2 \\
&= \int_{-\pi}^{\pi} |\sin\left[\left(N+\frac{1}{2}\right)\theta\right]|^2 d\theta \\
&= \int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos\left[(2N+1)\theta\right] d\theta \\
&= \left[\frac{\theta}{2}\right]_0^{2\pi} - \left[\frac{1}{2N+1}\sin\left(2N\theta+\theta\right)]_0^{2\pi}\right]_0^{2\pi} \\
&= \pi
\end{aligned}$$

So I now have:

$$\sum_{N=1}^{\infty} \frac{|g, f_N|^2}{\pi} \leqslant ||g||^2$$

If $||g||$ is finite, then by 2.9, the series in the left hand side of the Bessels inequality converges to zero, which is only possible if (g, f_N) tends to 0.

$$||g||^2 = \int_{-\pi}^{\pi} \frac{[f(x+\theta) - f(x)]^2}{sin^2(\frac{1}{2}\theta)} d\theta$$

is continuous everywhere except perhaps when $sin^2(\frac{1}{2}\theta) = 0$, which is the case when $\theta = 0$. However, by rewriting the integrand like so:

$$\frac{f(x+\theta) - f(x)}{\theta} \cdot \frac{\theta}{sin(\frac{1}{2}\theta},$$

I see that this is just the definition of the derivative f'(x), multiplied by $\frac{\theta}{\sin\left(\frac{1}{2}\theta\right)}$. Taking the limit as θ goes to 0, and making use of L'Hôpitals rule:

$$lim_{\theta\to 0} g(\theta) = lim_{\theta\to 0} \frac{f(x+\theta) - f(x)}{\theta} \cdot \frac{\theta}{sin(\frac{1}{2}\theta}$$

$$= f'(x)\left[\frac{0}{0}\right]$$

$$= f'(x)\lim_{\theta\to 0}\frac{1}{\frac{1}{2}\cos\left(\frac{1}{2}\theta\right)}$$

$$= 2f'(x)$$

Now since f is C^1, g is continuous so the L^2-norm of g is finite. So, I have shown that $S_N(x) - f(x)$ tends to 0 as N tends to ∞, which concludes the proof for C^1-functions.

In the case where f(x) and f'(x) are both only pointwisely continuous, meaning they are continuous except for a finite number of points at which the function has jump discontinuities.[5] Now, I want to show that the Fourier series converges even so. The steps in this proof are the same as in the previous case, however when subtracting f(x) from the sum, I now subtract by the sum by $\frac{1}{2}[f(x+) - f(x-)]$:

$$S_N - \frac{1}{2}[f(x+)+f(x-)] = \int_0^{\pi} K_N(\theta)[f(x+\theta)-f(x+)]\frac{d\theta}{2\pi} + \int_{-\pi}^{0} K_N(\theta)[f(x+\theta)-f(x-)]\frac{d\theta}{2\pi}$$

Remembering that $K_N(\theta) = \frac{\sin\left(N+\frac{1}{2}\right)}{\sin\left(\frac{1}{2}\theta\right)}$ (A.3), and defining $g_+(x) = \frac{f(x+\theta)-f(x+}{\sin\left(\frac{1}{2}\theta\right)}$ and $g_-(x) = \frac{f(x+\theta)-f(x-}{\sin\left(\frac{1}{2}\theta\right)}$ I get:

$$S_N - g_{\pm}(x) = \int_0^{\pi} g_+(\theta)sin[(N+\frac{1}{2})\theta]d\theta + \int_{-\pi}^{0} g_-(\theta)sin[(N+\frac{1}{2})\theta]d\theta$$

Now, I observe that $sin[(N+\frac{1}{2})\theta]$ forms an orthogonal set of functions on the intervals $(-\pi, 0)$ and $(0, \pi)$. As previously, Bessels inequality tells me that if $\int_0^{\pi} |g_+(\theta)|^2$ and $\int_0^{\pi} |g_-(\theta)|^2$ are finite,then:

$$\int_0^{\pi} g_+(\theta)sin[(N+\frac{1}{2})\theta]d\theta \xrightarrow{N\to\infty} 0$$

and

$$\int_{-\pi}^{0} g_-(\theta)sin[(N+\frac{1}{2})\theta]d\theta \xrightarrow{N\to\infty} 0$$

The only possible cause of divergence of $\int_0^{\pi} |g_+(\theta)|^2$ and $\int_{-\pi}^{0} |g_-(\theta)|^2$ is when the numerator of the $g_{\pm}(\theta)$ is zero, i.e. when $sin(\frac{1}{2}\theta) = 0$, which happens when $\theta = 0$.

However, when taking limit from the left gives me:

$$lim_{\theta \to 0^+} g_+(\theta) = lim_{\theta \to 0^+} g_+(\theta) = lim_{\theta \to 0^+} \frac{f(x+\theta)}{\theta} \cdot \frac{\theta}{sin(\frac{1}{2}\theta)} = 2f'(x+)$$

for some x where $f'(x+)$ exists. When this does not exist, f is still differentiable at points near x, by the mean value theorem. The derivative is bounded, and for the same reason the derivative $f'(x+)$ is bounded for certain value of θ near 0. As a result $g_+(\theta)$ is bounded and

$$||g_+(\theta)||$$

is finite. The same argument can be shown for $g_-(\theta)$. These two results imply pointwise convergence of the function f(x) with discontinuities. $\square$